edexcel

# Factorising expressions

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Factorising an expression is the opposite of expanding the brackets.
- A quadratic expression is in the form $ax^2 + bx + c$, where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is $b$ and whose product is $ac$.
- An expression in the form $x^2 - y^2$ is called the difference of two squares. It factorises to $(x - y)(x + y)$.

## Examples

**Example 1** Factorise $15x^2y^3 + 9x^4y$

| | |
|---|---|
| $15x^2y^3 + 9x^4y = 3x^2y(5y^2 + 3x^2)$ | The highest common factor is $3x^2y$. So take $3x^2y$ outside the brackets and then divide each term by $3x^2y$ to find the terms in the brackets |

**Example 2** Factorise $4x^2 - 25y^2$

| | |
|---|---|
| $4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$ | This is the difference of two squares as the two terms can be written as $(2x)^2$ and $(5y)^2$ |

**Example 3** Factorise $x^2 + 3x - 10$

| | |
|---|---|
| $b = 3$, $ac = -10$ | **1** Work out the two factors of $ac = -10$ which add to give $b = 3$ (5 and −2) |
| So $x^2 + 3x - 10 = x^2 + 5x - 2x - 10$ | **2** Rewrite the $b$ term ($3x$) using these two factors |
| $= x(x + 5) - 2(x + 5)$ | **3** Factorise the first two terms and the last two terms |
| $= (x + 5)(x - 2)$ | **4** $(x + 5)$ is a factor of both terms |

Pearson

**Example 4** Factorise $6x^2 - 11x - 10$

| | |
|---|---|
| $b = -11$, $ac = -60$<br><br>So<br>$6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$<br><br>$= 3x(2x - 5) + 2(2x - 5)$<br><br>$= (2x - 5)(3x + 2)$ | **1** Work out the two factors of $ac = -60$ which add to give $b = -11$ ($-15$ and 4)<br>**2** Rewrite the $b$ term ($-11x$) using these two factors<br>**3** Factorise the first two terms and the last two terms<br>**4** $(2x - 5)$ is a factor of both terms |

**Example 5** Simplify $\dfrac{x^2-4x-21}{2x^2+9x+9}$

| | |
|---|---|
| $\dfrac{x^2-4x-21}{2x^2+9x+9}$ | **1** Factorise the numerator and the denominator |
| For the numerator:<br>$b = -4$, $ac = -21$ | **2** Work out the two factors of $ac = -21$ which add to give $b = -4$ ($-7$ and 3) |
| So<br>$x^2 - 4x - 21 = x^2 - 7x + 3x - 21$<br><br>$= x(x - 7) + 3(x - 7)$<br><br>$= (x - 7)(x + 3)$ | **3** Rewrite the $b$ term ($-4x$) using these two factors<br>**4** Factorise the first two terms and the last two terms<br>**5** $(x - 7)$ is a factor of both terms |
| For the denominator:<br>$b = 9$, $ac = 18$ | **6** Work out the two factors of $ac = 18$ which add to give $b = 9$ (6 and 3) |
| So<br>$2x^2 + 9x + 9 = 2x^2 + 6x + 3x + 9$<br><br>$= 2x(x + 3) + 3(x + 3)$<br><br>$= (x + 3)(2x + 3)$ | **7** Rewrite the $b$ term ($9x$) using these two factors<br>**8** Factorise the first two terms and the last two terms<br>**9** $(x + 3)$ is a factor of both terms |
| So<br>$\dfrac{x^2-4x-21}{2x^2+9x+9} = \dfrac{(x-7)(x+3)}{(x+3)(2x+3)}$<br><br>$= \dfrac{x-7}{2x+3}$ | **10** $(x + 3)$ is a factor of both the numerator and denominator so cancels out as a value divided by itself is 1 |

# Practice

**1** Factorise.

**a** $6x^4y^3 - 10x^3y^4$ **b** $21a^3b^5 + 35a^5b^2$

**c** $25x^2y^2 - 10x^3y^2 + 15x^2y^3$

**Hint**

Take the highest common factor outside the bracket.

**2** Factorise

**a** $x^2 + 7x + 12$ **b** $x^2 + 5x - 14$

**c** $x^2 - 11x + 30$ **d** $x^2 - 5x - 24$

**e** $x^2 - 7x - 18$ **f** $x^2 + x - 20$

**g** $x^2 - 3x - 40$ **h** $x^2 + 3x - 28$

**3** Factorise

**a** $36x^2 - 49y^2$ **b** $4x^2 - 81y^2$

**c** $18a^2 - 200b^2c^2$

**4** Factorise

**a** $2x^2 + x - 3$ **b** $6x^2 + 17x + 5$

**c** $2x^2 + 7x + 3$ **d** $9x^2 - 15x + 4$

**e** $10x^2 + 21x + 9$ **f** $12x^2 - 38x + 20$

**5** Simplify the algebraic fractions.

**a** $\dfrac{2x^2 + 4x}{x^2 - x}$ **b** $\dfrac{x^2 + 3x}{x^2 + 2x - 3}$

**c** $\dfrac{x^2 - 2x - 8}{x^2 - 4x}$ **d** $\dfrac{x^2 - 5x}{x^2 - 25}$

**e** $\dfrac{x^2 - x - 12}{x^2 - 4x}$ **f** $\dfrac{2x^2 + 14x}{2x^2 + 4x - 70}$

**6** Simplify

**a** $\dfrac{9x^2 - 16}{3x^2 + 17x - 28}$ **b** $\dfrac{2x^2 - 7x - 15}{3x^2 - 17x + 10}$

**c** $\dfrac{4 - 25x^2}{10x^2 - 11x - 6}$ **d** $\dfrac{6x^2 - x - 1}{2x^2 + 7x - 4}$

# Extend

**7** Simplify $\sqrt{x^2 + 10x + 25}$

**8** Simplify $\dfrac{(x+2)^2 + 3(x+2)^2}{x^2 - 4}$

## Answers

**1** **a** $2x^3y^3(3x - 5y)$ **b** $7a^3b^2(3b^3 + 5a^2)$
**c** $5x^2y^2(5 - 2x + 3y)$

**2** **a** $(x + 3)(x + 4)$ **b** $(x + 7)(x - 2)$
**c** $(x - 5)(x - 6)$ **d** $(x - 8)(x + 3)$
**e** $(x - 9)(x + 2)$ **f** $(x + 5)(x - 4)$
**g** $(x - 8)(x + 5)$ **h** $(x + 7)(x - 4)$

**3** **a** $(6x - 7y)(6x + 7y)$ **b** $(2x - 9y)(2x + 9y)$
**c** $2(3a - 10bc)(3a + 10bc)$

**4** **a** $(x - 1)(2x + 3)$ **b** $(3x + 1)(2x + 5)$
**c** $(2x + 1)(x + 3)$ **d** $(3x - 1)(3x - 4)$
**e** $(5x + 3)(2x + 3)$ **f** $2(3x - 2)(2x - 5)$

**5** **a** $\dfrac{2(x+2)}{x-1}$ **b** $\dfrac{x}{x-1}$
**c** $\dfrac{x+2}{x}$ **d** $\dfrac{x}{x+5}$
**e** $\dfrac{x+3}{x}$ **f** $\dfrac{x}{x-5}$

**6** **a** $\dfrac{3x+4}{x+7}$ **b** $\dfrac{2x+3}{3x-2}$
**c** $\dfrac{2-5x}{2x-3}$ **d** $\dfrac{3x+1}{x+4}$

**7** $(x + 5)$

**8** $\dfrac{4(x+2)}{x-2}$

edexcel

# Completing the square

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using $a$ as a common factor.

## Examples

**Example 1** Complete the square for the quadratic expression $x^2 + 6x - 2$

| | |
|---|---|
| $x^2 + 6x - 2$<br><br>$= (x + 3)^2 - 9 - 2$<br><br>$= (x + 3)^2 - 11$ | **1** Write $x^2 + bx + c$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$<br><br>**2** Simplify |

**Example 2** Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

| | |
|---|---|
| $2x^2 - 5x + 1$<br><br>$= 2\left(x^2 - \frac{5}{2}x\right) + 1$<br><br>$= 2\left[\left(x-\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$<br><br>$= 2\left(x-\frac{5}{4}\right)^2 - \frac{25}{8} + 1$<br><br>$= 2\left(x-\frac{5}{4}\right)^2 - \frac{17}{8}$ | **1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$<br><br>**2** Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$<br><br>**3** Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2<br><br>**4** Simplify |

Pearson

## Practice

**1** Write the following quadratic expressions in the form $(x + p)^2 + q$

| | | | |
|---|---|---|---|
| **a** | $x^2 + 4x + 3$ | **b** | $x^2 – 10x – 3$ |
| **c** | $x^2 – 8x$ | **d** | $x^2 + 6x$ |
| **e** | $x^2 – 2x + 7$ | **f** | $x^2 + 3x – 2$ |

**2** Write the following quadratic expressions in the form $p(x + q)^2 + r$

| | | | |
|---|---|---|---|
| **a** | $2x^2 – 8x – 16$ | **b** | $4x^2 – 8x – 16$ |
| **c** | $3x^2 + 12x – 9$ | **d** | $2x^2 + 6x – 8$ |

**3** Complete the square.

| | | | |
|---|---|---|---|
| **a** | $2x^2 + 3x + 6$ | **b** | $3x^2 – 2x$ |
| **c** | $5x^2 + 3x$ | **d** | $3x^2 + 5x + 3$ |

## Extend

**4** Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

## Answers

**1** **a** $(x+2)^2-1$ **b** $(x-5)^2-28$

**c** $(x-4)^2-16$ **d** $(x+3)^2-9$

**e** $(x-1)^2+6$ **f** $\left(x+\frac{3}{2}\right)^2-\frac{17}{4}$

**2** **a** $2(x-2)^2-24$ **b** $4(x-1)^2-20$

**c** $3(x+2)^2-21$ **d** $2\left(x+\frac{3}{2}\right)^2-\frac{25}{2}$

**3** **a** $2\left(x+\frac{3}{4}\right)^2+\frac{39}{8}$ **b** $3\left(x-\frac{1}{3}\right)^2-\frac{1}{3}$

**c** $5\left(x+\frac{3}{10}\right)^2-\frac{9}{20}$ **d** $3\left(x+\frac{5}{6}\right)^2+\frac{11}{12}$

**4** $(5x+3)^2+3$

# Solving quadratic equations by factorisation

| |
|---|
| **A LEVEL LINKS**<br>**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants |

## Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is $b$ and whose products is $ac$.
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

## Examples

**Example 1** Solve $5x^2 = 15x$

| | |
|---|---|
| $5x^2 = 15x$<br><br>$5x^2 - 15x = 0$ | **1** Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by $x$ as this would lose the solution $x = 0$. |
| $5x(x - 3) = 0$ | **2** Factorise the quadratic equation. $5x$ is a common factor. |
| So $5x = 0$ or $(x - 3) = 0$ | **3** When two values multiply to make zero, at least one of the values must be zero. |
| Therefore $x = 0$ or $x = 3$ | **4** Solve these two equations. |

**Example 2** Solve $x^2 + 7x + 12 = 0$

| | |
|---|---|
| $x^2 + 7x + 12 = 0$<br><br>$b = 7$, $ac = 12$ | **1** Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3) |
| $x^2 + 4x + 3x + 12 = 0$ | **2** Rewrite the $b$ term ($7x$) using these two factors. |
| $x(x + 4) + 3(x + 4) = 0$ | **3** Factorise the first two terms and the last two terms. |
| $(x + 4)(x + 3) = 0$ | **4** $(x + 4)$ is a factor of both terms. |
| So $(x + 4) = 0$ or $(x + 3) = 0$ | **5** When two values multiply to make zero, at least one of the values must be zero. |
| Therefore $x = -4$ or $x = -3$ | **6** Solve these two equations. |

**Example 3** Solve $9x^2 - 16 = 0$

| | |
|---|---|
| $9x^2 - 16 = 0$<br>$(3x + 4)(3x - 4) = 0$<br><br>So $(3x + 4) = 0$ or $(3x - 4) = 0$<br><br>$x = -\frac{4}{3}$ or $x = \frac{4}{3}$ | **1** Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$.<br>**2** When two values multiply to make zero, at least one of the values must be zero.<br>**3** Solve these two equations. |

**Example 4** Solve $2x^2 - 5x - 12 = 0$

| | |
|---|---|
| $b = -5$, $ac = -24$<br><br>So $2x^2 - 8x + 3x - 12 = 0$<br><br>$2x(x - 4) + 3(x - 4) = 0$<br><br>$(x - 4)(2x + 3) = 0$<br>So $(x - 4) = 0$ or $(2x + 3) = 0$<br><br>$x = 4$ or $x = -\frac{3}{2}$ | **1** Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. ($-8$ and 3)<br>**2** Rewrite the $b$ term ($-5x$) using these two factors.<br>**3** Factorise the first two terms and the last two terms.<br>**4** $(x - 4)$ is a factor of both terms.<br>**5** When two values multiply to make zero, at least one of the values must be zero.<br>**6** Solve these two equations. |

## Practice

**1** Solve

**a** $6x^2 + 4x = 0$
**b** $28x^2 - 21x = 0$
**c** $x^2 + 7x + 10 = 0$
**d** $x^2 - 5x + 6 = 0$
**e** $x^2 - 3x - 4 = 0$
**f** $x^2 + 3x - 10 = 0$
**g** $x^2 - 10x + 24 = 0$
**h** $x^2 - 36 = 0$
**i** $x^2 + 3x - 28 = 0$
**j** $x^2 - 6x + 9 = 0$
**k** $2x^2 - 7x - 4 = 0$
**l** $3x^2 - 13x - 10 = 0$

**2** Solve

**a** $x^2 - 3x = 10$
**b** $x^2 - 3 = 2x$
**c** $x^2 + 5x = 24$
**d** $x^2 - 42 = x$
**e** $x(x + 2) = 2x + 25$
**f** $x^2 - 30 = 3x - 2$
**g** $x(3x + 1) = x^2 + 15$
**h** $3x(x - 1) = 2(x + 1)$

**Hint**

Get all terms onto one side of the equation.

# Solving quadratic equations by completing the square

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

## Examples

**Example 5** Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

| | |
|---|---|
| $x^2 + 6x + 4 = 0$ | **1** Write $x^2 + bx + c = 0$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0$ |
| $(x + 3)^2 - 9 + 4 = 0$ | |
| $(x + 3)^2 - 5 = 0$ | **2** Simplify. |
| $(x + 3)^2 = 5$ | **3** Rearrange the equation to work out $x$. First, add 5 to both sides. |
| $x + 3 = \pm\sqrt{5}$ | **4** Square root both sides. Remember that the square root of a value gives two answers. |
| $x = \pm\sqrt{5} - 3$ | **5** Subtract 3 from both sides to solve the equation. |
| So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$ | **6** Write down both solutions. |

**Example 6** Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

| | |
|---|---|
| $2x^2 - 7x + 4 = 0$ | **1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2+\frac{b}{a}x\right)+c$ |
| $2\left(x^2 - \frac{7}{2}x\right) + 4 = 0$ | |
| $2\left[\left(x-\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right] + 4 = 0$ | **2** Now complete the square by writing $x^2 - \frac{7}{2}x$ in the form $\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$ |
| $2\left(x-\frac{7}{4}\right)^2 - \frac{49}{8} + 4 = 0$ | **3** Expand the square brackets. |
| $2\left(x-\frac{7}{4}\right)^2 - \frac{17}{8} = 0$ | **4** Simplify. |

*(continued on next page)*

| | |
|---|---|
| $2\left(x-\frac{7}{4}\right)^2=\frac{17}{8}$ | **5** Rearrange the equation to work out $x$. First, add $\frac{17}{8}$ to both sides. |
| $\left(x-\frac{7}{4}\right)^2=\frac{17}{16}$ | **6** Divide both sides by 2. |
| $x-\frac{7}{4}=\pm\frac{\sqrt{17}}{4}$ | **7** Square root both sides. Remember that the square root of a value gives two answers. |
| $x=\pm\frac{\sqrt{17}}{4}+\frac{7}{4}$ | **8** Add $\frac{7}{4}$ to both sides. |
| So $x=\frac{7}{4}-\frac{\sqrt{17}}{4}$ or $x=\frac{7}{4}+\frac{\sqrt{17}}{4}$ | **9** Write down both the solutions. |

## Practice

**3** Solve by completing the square.

**a** $x^2 - 4x - 3 = 0$ **b** $x^2 - 10x + 4 = 0$

**c** $x^2 + 8x - 5 = 0$ **d** $x^2 - 2x - 6 = 0$

**e** $2x^2 + 8x - 5 = 0$ **f** $5x^2 + 3x - 4 = 0$

**4** Solve by completing the square.

**a** $(x - 4)(x + 2) = 5$

**b** $2x^2 + 6x - 7 = 0$

**c** $x^2 - 5x + 3 = 0$

**Hint**

Get all terms onto one side of the equation.

# Solving quadratic equations by using the formula

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for $a$, $b$ and $c$.

## Examples

**Example 7** Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

| | |
|---|---|
| $a = 1$, $b = 6$, $c = 4$<br>$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | **1** Identify $a$, $b$ and $c$ and write down the formula.<br>Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. |
| $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$ | **2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula. |
| $x = \dfrac{-6 \pm \sqrt{20}}{2}$ | **3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2. |
| $x = \dfrac{-6 \pm 2\sqrt{5}}{2}$ | **4** Simplify $\sqrt{20}$.<br>$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$ |
| $x = -3 \pm \sqrt{5}$ | **5** Simplify by dividing numerator and denominator by 2. |
| So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$ | **6** Write down both the solutions. |

**Example 8** Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

| | |
|---|---|
| $a = 3$, $b = -7$, $c = -2$ <br> $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | **1** Identify $a$, $b$ and $c$, making sure you get the signs right and write down the formula. <br> Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. |
| $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$ | **2** Substitute $a = 3$, $b = -7$, $c = -2$ into the formula. |
| $x = \dfrac{7 \pm \sqrt{73}}{6}$ | **3** Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2. |
| So $x = \dfrac{7 - \sqrt{73}}{6}$ or $x = \dfrac{7 + \sqrt{73}}{6}$ | **4** Write down both the solutions. |

## Practice

**5** Solve, giving your solutions in surd form.

**a** $3x^2 + 6x + 2 = 0$ **b** $2x^2 - 4x - 7 = 0$

**6** Solve the equation $x^2 - 7x + 2 = 0$

Give your solutions in the form $\dfrac{a \pm \sqrt{b}}{c}$, where $a$, $b$ and $c$ are integers.

**7** Solve $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

> **Hint**
>
> Get all terms onto one side of the equation.

## Extend

**8** Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

**a** $4x(x - 1) = 3x - 2$

**b** $10 = (x + 1)^2$

**c** $x(3x - 1) = 10$

## Answers

**1** **a** $x = 0$ or $x = -\frac{2}{3}$ **b** $x = 0$ or $x = \frac{3}{4}$

**c** $x = -5$ or $x = -2$ **d** $x = 2$ or $x = 3$

**e** $x = -1$ or $x = 4$ **f** $x = -5$ or $x = 2$

**g** $x = 4$ or $x = 6$ **h** $x = -6$ or $x = 6$

**i** $x = -7$ or $x = 4$ **j** $x = 3$

**k** $x = -\frac{1}{2}$ or $x = 4$ **l** $x = -\frac{2}{3}$ or $x = 5$

**2** **a** $x = -2$ or $x = 5$ **b** $x = -1$ or $x = 3$

**c** $x = -8$ or $x = 3$ **d** $x = -6$ or $x = 7$

**e** $x = -5$ or $x = 5$ **f** $x = -4$ or $x = 7$

**g** $x = -3$ or $x = 2\frac{1}{2}$ **h** $x = -\frac{1}{3}$ or $x = 2$

**3** **a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ **b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$

**c** $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$ **d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$

**e** $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$ **f** $x = \frac{-3+\sqrt{89}}{10}$ or $x = \frac{-3-\sqrt{89}}{10}$

**4** **a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$ **b** $x = \frac{-3+\sqrt{23}}{2}$ or $x = \frac{-3-\sqrt{23}}{2}$

**c** $x = \frac{5+\sqrt{13}}{2}$ or $x = \frac{5-\sqrt{13}}{2}$

**5** **a** $x = -1 + \frac{\sqrt{3}}{3}$ or $x = -1 - \frac{\sqrt{3}}{3}$ **b** $x = 1 + \frac{3\sqrt{2}}{2}$ or $x = 1 - \frac{3\sqrt{2}}{2}$

**6** $x = \frac{7+\sqrt{41}}{2}$ or $x = \frac{7-\sqrt{41}}{2}$

**7** $x = \frac{-3+\sqrt{89}}{20}$ or $x = \frac{-3-\sqrt{89}}{20}$

**8** **a** $x = \frac{7+\sqrt{17}}{8}$ or $x = \frac{7-\sqrt{17}}{8}$

**b** $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

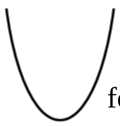
**c** $x = -1\frac{2}{3}$ or $x = 2$

# Sketching quadratic graphs

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- The graph of the quadratic function $y = ax^2 + bx + c$, where $a \neq 0$, is a curve called a parabola.
- Parabolas have a line of symmetry and a shape as shown.

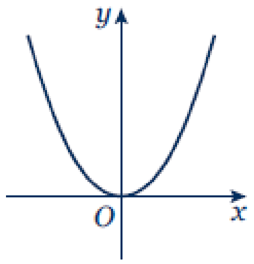

for $a > 0$


for $a < 0$

- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the $y$-axis substitute $x = 0$ into the function.
- To find where the curve intersects the $x$-axis substitute $y = 0$ into the function.
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- To find the coordinates of the maximum or minimum point (turning points) of a quadratic curve (parabola) you can use the completed square form of the function.

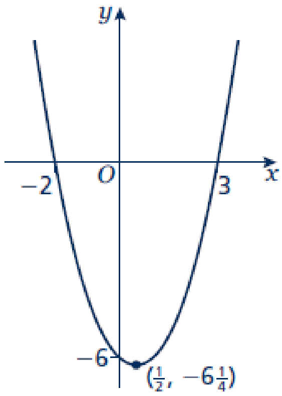
## Examples

**Example 1** Sketch the graph of $y = x^2$.

| | |
|---|---|
|  | The graph of $y = x^2$ is a parabola.<br><br>When $x = 0$, $y = 0$.<br><br>$a = 1$ which is greater than zero, so the graph has the shape:  |

**Example 2** Sketch the graph of $y = x^2 - x - 6$.

| | |
|---|---|
| When $x = 0$, $y = 0^2 - 0 - 6 = -6$<br>So the graph intersects the $y$-axis at $(0, -6)$ | **1** Find where the graph intersects the $y$-axis by substituting $x = 0$. |
| When $y = 0$, $x^2 - x - 6 = 0$ | **2** Find where the graph intersects the $x$-axis by substituting $y = 0$. |
| $(x + 2)(x - 3) = 0$ | **3** Solve the equation by factorising. |
| $x = -2$ or $x = 3$ | **4** Solve $(x + 2) = 0$ and $(x - 3) = 0$. |
| So,<br>the graph intersects the $x$-axis at $(-2, 0)$ and $(3, 0)$ | **5** $a = 1$ which is greater than zero, so the graph has the shape:  |
| | *(continued on next page)* |
| | **6** To find the turning point, complete |

| | |
|---|---|
| $x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 6$ <br> $= \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$ <br> When $\left(x - \frac{1}{2}\right)^2 = 0$, $x = \frac{1}{2}$ and <br> $y = -\frac{25}{4}$, so the turning point is at the point $\left(\frac{1}{2}, -\frac{25}{4}\right)$ <br>   | the square. <br><br> **7** The turning point is the minimum value for this expression and occurs when the term in the bracket is equal to zero. |

## Practice

**1** Sketch the graph of $y = -x^2$.

**2** Sketch each graph, labelling where the curve crosses the axes.

**a** $y = (x + 2)(x - 1)$ **b** $y = x(x - 3)$ **c** $y = (x + 1)(x + 5)$

**3** Sketch each graph, labelling where the curve crosses the axes.

**a** $y = x^2 - x - 6$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 - 4$

**d** $y = x^2 + 4x$ **e** $y = 9 - x^2$ **f** $y = x^2 + 2x - 3$

**4** Sketch the graph of $y = 2x^2 + 5x - 3$, labelling where the curve crosses the axes.

## Extend

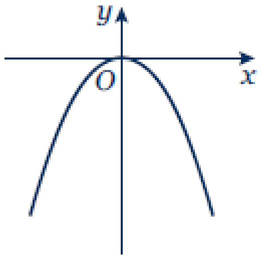
**5** Sketch each graph. Label where the curve crosses the axes and write down the coordinates of the turning point.

**a** $y = x^2 - 5x + 6$ **b** $y = -x^2 + 7x - 12$ **c** $y = -x^2 + 4x$

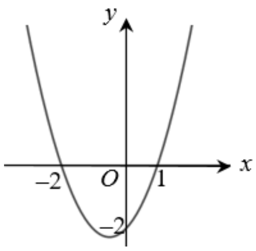
**6** Sketch the graph of $y = x^2 + 2x + 1$. Label where the curve crosses the axes and write down the equation of the line of symmetry.

## Answers

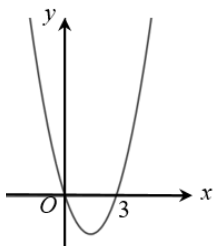
**1**



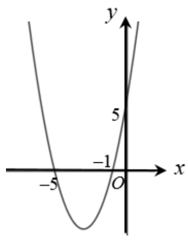

**2** **a**



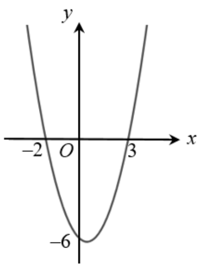

**b**



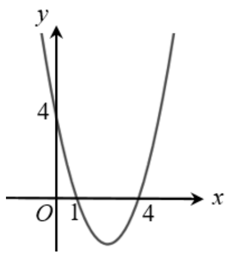

**c**



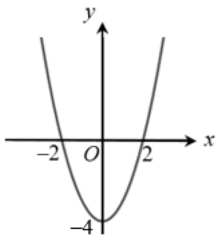

**3** **a**



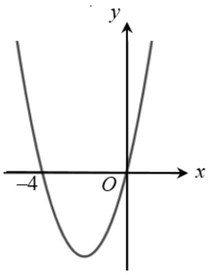

**b**



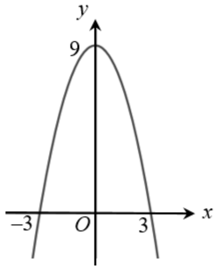

**c**



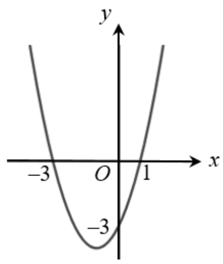

**d**



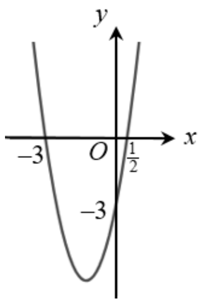

**e**



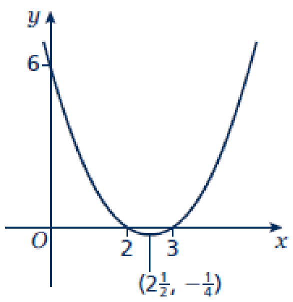

**f**



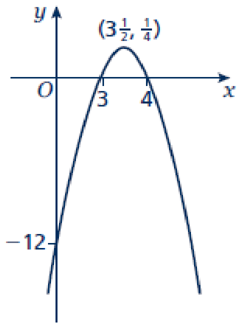
**4**



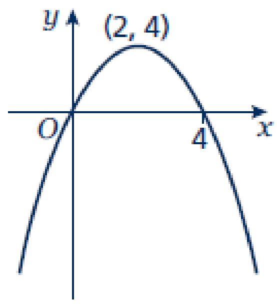

**5** **a**



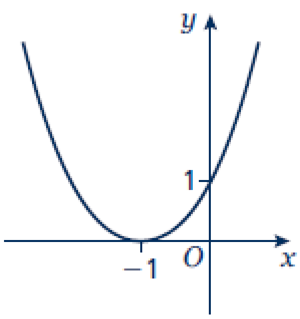

**b**




**c**




**6**




Line of symmetry at $x = -1$.

# Expanding brackets and simplifying expressions

**A LEVEL LINKS**
**Scheme of work:** 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

## Key points

- When you expand one set of brackets you must multiply everything inside the bracket by what is outside.
- When you expand two linear expressions, each with two terms of the form $ax + b$, where $a \neq 0$ and $b \neq 0$, you create four terms. Two of these can usually be simplified by collecting like terms.

## Examples

**Example 1** Expand $4(3x - 2)$

| | |
|---|---|
| $4(3x - 2) = 12x - 8$ | Multiply everything inside the bracket by the 4 outside the bracket |

**Example 2** Expand and simplify $3(x + 5) - 4(2x + 3)$

| | |
|---|---|
| $3(x + 5) - 4(2x + 3)$<br>$= 3x + 15 - 8x - 12$<br><br>$= 3 - 5x$ | **1** Expand each set of brackets separately by multiplying $(x + 5)$ by 3 and $(2x + 3)$ by $-4$<br><br>**2** Simplify by collecting like terms: $3x - 8x = -5x$ and $15 - 12 = 3$ |

**Example 3** Expand and simplify $(x + 3)(x + 2)$

| | |
|---|---|
| $(x + 3)(x + 2)$<br>$= x(x + 2) + 3(x + 2)$<br>$= x^2 + 2x + 3x + 6$<br>$= x^2 + 5x + 6$ | **1** Expand the brackets by multiplying $(x + 2)$ by $x$ and $(x + 2)$ by 3<br><br>**2** Simplify by collecting like terms: $2x + 3x = 5x$ |

**Example 4** Expand and simplify $(x - 5)(2x + 3)$

| | |
|---|---|
| $(x - 5)(2x + 3)$<br>$= x(2x + 3) - 5(2x + 3)$<br>$= 2x^2 + 3x - 10x - 15$<br>$= 2x^2 - 7x - 15$ | **1** Expand the brackets by multiplying $(2x + 3)$ by $x$ and $(2x + 3)$ by $-5$<br><br>**2** Simplify by collecting like terms: $3x - 10x = -7x$ |

## Practice

**Watch out!**

When multiplying (or dividing) positive and negative numbers, if the signs are the same the answer is '+'; if the signs are different the answer is '–'.

**1** Expand.

**a** $3(2x - 1)$  **b** $-2(5pq + 4q^2)$

**c** $-(3xy - 2y^2)$

**2** Expand and simplify.

**a** $7(3x + 5) + 6(2x - 8)$  **b** $8(5p - 2) - 3(4p + 9)$

**c** $9(3s + 1) -5(6s - 10)$  **d** $2(4x - 3) - (3x + 5)$

**3** Expand.

**a** $3x(4x + 8)$  **b** $4k(5k^2 - 12)$

**c** $-2h(6h^2 + 11h - 5)$  **d** $-3s(4s^2 - 7s + 2)$

**4** Expand and simplify.

**a** $3(y^2 - 8) - 4(y^2 - 5)$  **b** $2x(x + 5) + 3x(x - 7)$

**c** $4p(2p - 1) - 3p(5p - 2)$  **d** $3b(4b - 3) - b(6b - 9)$

**5** Expand $\frac{1}{2}(2y - 8)$

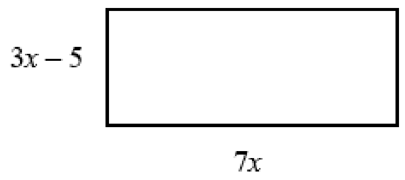
**6** Expand and simplify.

**a** $13 - 2(m + 7)$  **b** $5p(p^2 + 6p) - 9p(2p - 3)$

**7** The diagram shows a rectangle.

Write down an expression, in terms of $x$, for the area of the rectangle.

Show that the area of the rectangle can be written as $21x^2 - 35x$




**8** Expand and simplify.

**a** $(x + 4)(x + 5)$  **b** $(x + 7)(x + 3)$

**c** $(x + 7)(x - 2)$  **d** $(x + 5)(x - 5)$

**e** $(2x + 3)(x - 1)$  **f** $(3x - 2)(2x + 1)$

**g** $(5x - 3)(2x - 5)$  **h** $(3x - 2)(7 + 4x)$

**i** $(3x + 4y)(5y + 6x)$  **j** $(x + 5)^2$

**k** $(2x - 7)^2$  **l** $(4x - 3y)^2$

## Extend

**9** Expand and simplify $(x + 3)^2 + (x - 4)^2$

**10** Expand and simplify.

**a** $\left(x+\frac{1}{x}\right)\left(x-\frac{2}{x}\right)$  **b** $\left(x+\frac{1}{x}\right)^2$

## Answers

**1** **a** $6x - 3$ **b** $-10pq - 8q^2$
**c** $-3xy + 2y^2$

**2** **a** $21x + 35 + 12x - 48 = 33x - 13$
**b** $40p - 16 - 12p - 27 = 28p - 43$
**c** $27s + 9 - 30s + 50 = -3s + 59 = 59 - 3s$
**d** $8x - 6 - 3x - 5 = 5x - 11$

**3** **a** $12x^2 + 24x$ **b** $20k^3 - 48k$
**c** $10h - 12h^3 - 22h^2$ **d** $21s^2 - 21s^3 - 6s$

**4** **a** $-y^2 - 4$ **b** $5x^2 - 11x$
**c** $2p - 7p^2$ **d** $6b^2$

**5** $y - 4$

**6** **a** $-1 - 2m$ **b** $5p^3 + 12p^2 + 27p$

**7** $7x(3x - 5) = 21x^2 - 35x$

**8** **a** $x^2 + 9x + 20$ **b** $x^2 + 10x + 21$
**c** $x^2 + 5x - 14$ **d** $x^2 - 25$
**e** $2x^2 + x - 3$ **f** $6x^2 - x - 2$
**g** $10x^2 - 31x + 15$ **h** $12x^2 + 13x - 14$
**i** $18x^2 + 39xy + 20y^2$ **j** $x^2 + 10x + 25$
**k** $4x^2 - 28x + 49$ **l** $16x^2 - 24xy + 9y^2$

**9** $2x^2 - 2x + 25$

**10** **a** $x^2 - 1 - \frac{2}{x^2}$ **b** $x^2 + 2 + \frac{1}{x^2}$

# Surds and rationalising the denominator

**A LEVEL LINKS**
**Scheme of work:** 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

## Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5},$ etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\frac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd $\sqrt{b}$
- To rationalise $\frac{a}{b+\sqrt{c}}$ you multiply the numerator and denominator by $b-\sqrt{c}$

## Examples

**Example 1** Simplify $\sqrt{50}$

| | |
|---|---|
| $\sqrt{50} = \sqrt{25 \times 2}$ | **1** Choose two numbers that are factors of 50. One of the factors must be a square number |
| $= \sqrt{25} \times \sqrt{2}$ | **2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ |
| $= 5 \times \sqrt{2}$<br>$= 5\sqrt{2}$ | **3** Use $\sqrt{25} = 5$ |

**Example 2** Simplify $\sqrt{147} - 2\sqrt{12}$

| | |
|---|---|
| $\sqrt{147} - 2\sqrt{12}$<br>$= \sqrt{49 \times 3} - 2\sqrt{4 \times 3}$ | **1** Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number |
| $= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3}$ | **2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ |
| $= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3}$ | **3** Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$ |
| $= 7\sqrt{3} - 4\sqrt{3}$<br>$= 3\sqrt{3}$ | **4** Collect like terms |

**Example 3** Simplify $\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$

| | |
|---|---|
| $\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$<br>$= \sqrt{49}-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}-\sqrt{4}$<br><br>$= 7 - 2$<br>$= 5$ | **1** Expand the brackets. A common mistake here is to write $\left(\sqrt{7}\right)^2 = 49$<br><br>**2** Collect like terms:<br>$-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}$<br>$=-\sqrt{7}\sqrt{2}+\sqrt{7}\sqrt{2}=0$ |

**Example 4** Rationalise $\dfrac{1}{\sqrt{3}}$

| | |
|---|---|
| $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$<br><br>$= \dfrac{1\times\sqrt{3}}{\sqrt{9}}$<br><br>$= \dfrac{\sqrt{3}}{3}$ | **1** Multiply the numerator and denominator by $\sqrt{3}$<br><br>**2** Use $\sqrt{9}=3$ |

**Example 5** Rationalise and simplify $\dfrac{\sqrt{2}}{\sqrt{12}}$

| | |
|---|---|
| $\dfrac{\sqrt{2}}{\sqrt{12}} = \dfrac{\sqrt{2}}{\sqrt{12}} \times \dfrac{\sqrt{12}}{\sqrt{12}}$<br><br>$= \dfrac{\sqrt{2}\times\sqrt{4\times3}}{12}$<br><br>$= \dfrac{2\sqrt{2}\sqrt{3}}{12}$<br><br>$= \dfrac{\sqrt{2}\sqrt{3}}{6}$ | **1** Multiply the numerator and denominator by $\sqrt{12}$<br><br>**2** Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number<br><br>**3** Use the rule $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$<br>**4** Use $\sqrt{4}=2$<br><br>**5** Simplify the fraction: $\dfrac{2}{12}$ simplifies to $\dfrac{1}{6}$ |

**Example 6** Rationalise and simplify $\dfrac{3}{2+\sqrt{5}}$

| | |
|---|---|
| $\dfrac{3}{2+\sqrt{5}} = \dfrac{3}{2+\sqrt{5}} \times \dfrac{2-\sqrt{5}}{2-\sqrt{5}}$ | **1** Multiply the numerator and denominator by $2-\sqrt{5}$ |
| $= \dfrac{3\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}$ | **2** Expand the brackets |
| $= \dfrac{6-3\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-5}$ | **3** Simplify the fraction |
| $= \dfrac{6-3\sqrt{5}}{-1}$ | **4** Divide the numerator by $-1$ Remember to change the sign of all terms when dividing by $-1$ |
| $= 3\sqrt{5}-6$ | |

## Practice

**1** Simplify.

**a** $\sqrt{45}$ **b** $\sqrt{125}$

**c** $\sqrt{48}$ **d** $\sqrt{175}$

**e** $\sqrt{300}$ **f** $\sqrt{28}$

**g** $\sqrt{72}$ **h** $\sqrt{162}$

**Hint**

One of the two numbers you choose at the start must be a square number.

**2** Simplify.

**a** $\sqrt{72}+\sqrt{162}$ **b** $\sqrt{45}-2\sqrt{5}$

**c** $\sqrt{50}-\sqrt{8}$ **d** $\sqrt{75}-\sqrt{48}$

**e** $2\sqrt{28}+\sqrt{28}$ **f** $2\sqrt{12}-\sqrt{12}+\sqrt{27}$

**Watch out!**

Check you have chosen the highest square number at the start.

**3** Expand and simplify.

**a** $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **b** $(3+\sqrt{3})(5-\sqrt{12})$

**c** $(4-\sqrt{5})(\sqrt{45}+2)$ **d** $(5+\sqrt{2})(6-\sqrt{8})$

**4** Rationalise and simplify, if possible.

**a** $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$

**c** $\frac{2}{\sqrt{7}}$ **d** $\frac{2}{\sqrt{8}}$

**e** $\frac{2}{\sqrt{2}}$ **f** $\frac{5}{\sqrt{5}}$

**g** $\frac{\sqrt{8}}{\sqrt{24}}$ **h** $\frac{\sqrt{5}}{\sqrt{45}}$

**5** Rationalise and simplify.

**a** $\frac{1}{3-\sqrt{5}}$ **b** $\frac{2}{4+\sqrt{3}}$ **c** $\frac{6}{5-\sqrt{2}}$

## Extend

**6** Expand and simplify $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)$

**7** Rationalise and simplify, if possible.

**a** $\frac{1}{\sqrt{9}-\sqrt{8}}$ **b** $\frac{1}{\sqrt{x}-\sqrt{y}}$

## Answers

**1** **a** $3\sqrt{5}$ **b** $5\sqrt{5}$

**c** $4\sqrt{3}$ **d** $5\sqrt{7}$

**e** $10\sqrt{3}$ **f** $2\sqrt{7}$

**g** $6\sqrt{2}$ **h** $9\sqrt{2}$

**2** **a** $15\sqrt{2}$ **b** $\sqrt{5}$

**c** $3\sqrt{2}$ **d** $\sqrt{3}$

**e** $6\sqrt{7}$ **f** $5\sqrt{3}$

**3** **a** $-1$ **b** $9-\sqrt{3}$

**c** $10\sqrt{5}-7$ **d** $26-4\sqrt{2}$

**4** **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{11}}{11}$

**c** $\frac{2\sqrt{7}}{7}$ **d** $\frac{\sqrt{2}}{2}$

**e** $\sqrt{2}$ **f** $\sqrt{5}$

**g** $\frac{\sqrt{3}}{3}$ **h** $\frac{1}{3}$

**5** **a** $\frac{3+\sqrt{5}}{4}$ **b** $\frac{2(4-\sqrt{3})}{13}$ **c** $\frac{6(5+\sqrt{2})}{23}$

**6** $x-y$

**7** **a** $3+2\sqrt{2}$ **b** $\frac{\sqrt{x}+\sqrt{y}}{x-y}$

# Rules of indices

**A LEVEL LINKS**
**Scheme of work:** 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

## Key points

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the $n$th root of $a$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \dfrac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

## Examples

**Example 1** Evaluate $10^0$

| | |
|---|---|
| $10^0 = 1$ | Any value raised to the power of zero is equal to 1 |

**Example 2** Evaluate $9^{\frac{1}{2}}$

| | |
|---|---|
| $9^{\frac{1}{2}} = \sqrt{9}$<br>$= 3$ | Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$ |

**Example 3** Evaluate $27^{\frac{2}{3}}$

| | |
|---|---|
| $27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$<br>$= 3^2$<br>$= 9$ | **1** Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$<br>**2** Use $\sqrt[3]{27} = 3$ |

**Example 4** Evaluate $4^{-2}$

| | |
|---|---|
| $4^{-2} = \frac{1}{4^2}$<br>$= \frac{1}{16}$ | **1** Use the rule $a^{-m} = \frac{1}{a^m}$<br>**2** Use $4^2 = 16$ |

**Example 5** Simplify $\frac{6x^5}{2x^2}$

| | |
|---|---|
| $\frac{6x^5}{2x^2} = 3x^3$ | $6 \div 2 = 3$ and use the rule $\frac{a^m}{a^n} = a^{m-n}$ to give $\frac{x^5}{x^2} = x^{5-2} = x^3$ |

**Example 6** Simplify $\frac{x^3 \times x^5}{x^4}$

| | |
|---|---|
| $\frac{x^3 \times x^5}{x^4} = \frac{x^{3+5}}{x^4} = \frac{x^8}{x^4}$<br>$= x^{8-4} = x^4$ | **1** Use the rule $a^m \times a^n = a^{m+n}$<br>**2** Use the rule $\frac{a^m}{a^n} = a^{m-n}$ |

**Example 7** Write $\frac{1}{3x}$ as a single power of $x$

| | |
|---|---|
| $\frac{1}{3x} = \frac{1}{3}x^{-1}$ | Use the rule $\frac{1}{a^m} = a^{-m}$, note that the fraction $\frac{1}{3}$ remains unchanged |

**Example 8** Write $\frac{4}{\sqrt{x}}$ as a single power of $x$

| | |
|---|---|
| $\frac{4}{\sqrt{x}} = \frac{4}{x^{\frac{1}{2}}}$<br>$= 4x^{-\frac{1}{2}}$ | **1** Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$<br>**2** Use the rule $\frac{1}{a^m} = a^{-m}$ |

## Practice

**1** Evaluate.

**a** $14^0$ **b** $3^0$ **c** $5^0$ **d** $x^0$

**2** Evaluate.

**a** $49^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $125^{\frac{1}{3}}$ **d** $16^{\frac{1}{4}}$

**3** Evaluate.

**a** $25^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $16^{\frac{3}{4}}$

**4** Evaluate.

**a** $5^{-2}$ **b** $4^{-3}$ **c** $2^{-5}$ **d** $6^{-2}$

**5** Simplify.

**a** $\dfrac{3x^2 \times x^3}{2x^2}$ **b** $\dfrac{10x^5}{2x^2 \times x}$

**c** $\dfrac{3x \times 2x^3}{2x^3}$ **d** $\dfrac{7x^3y^2}{14x^5y}$

**e** $\dfrac{y^2}{y^{\frac{1}{2}} \times y}$ **f** $\dfrac{c^{\frac{1}{2}}}{c^2 \times c^{\frac{3}{2}}}$

**g** $\dfrac{(2x^2)^3}{4x^0}$ **h** $\dfrac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^{-2} \times x^3}$

**Watch out!**

Remember that any value raised to the power of zero is 1. This is the rule $a^0 = 1$.

**6** Evaluate.

**a** $4^{-\frac{1}{2}}$ **b** $27^{-\frac{2}{3}}$ **c** $9^{-\frac{1}{2}} \times 2^3$

**d** $16^{\frac{1}{4}} \times 2^{-3}$ **e** $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}$ **f** $\left(\dfrac{27}{64}\right)^{-\frac{2}{3}}$

**7** Write the following as a single power of $x$.

**a** $\dfrac{1}{x}$ **b** $\dfrac{1}{x^7}$ **c** $\sqrt[4]{x}$

**d** $\sqrt[5]{x^2}$ **e** $\dfrac{1}{\sqrt[3]{x}}$ **f** $\dfrac{1}{\sqrt[3]{x^2}}$

8 Write the following without negative or fractional powers.

**a** $x^{-3}$ **b** $x^{0}$ **c** $x^{\frac{1}{5}}$

**d** $x^{\frac{2}{5}}$ **e** $x^{-\frac{1}{2}}$ **f** $x^{-\frac{3}{4}}$

9 Write the following in the form $ax^n$.

**a** $5\sqrt{x}$ **b** $\dfrac{2}{x^3}$ **c** $\dfrac{1}{3x^4}$

**d** $\dfrac{2}{\sqrt{x}}$ **e** $\dfrac{4}{\sqrt[3]{x}}$ **f** $3$

# Extend

10 Write as sums of powers of $x$.

**a** $\dfrac{x^5+1}{x^2}$ **b** $x^2\left(x+\dfrac{1}{x}\right)$ **c** $x^{-4}\left(x^2+\dfrac{1}{x^3}\right)$

edexcel

## Answers

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **1** | **a** | 1 | **b** | 1 | **c** | 1 | **d** | 1 |
| **2** | **a** | 7 | **b** | 4 | **c** | 5 | **d** | 2 |
| **3** | **a** | 125 | **b** | 32 | **c** | 343 | **d** | 8 |
| **4** | **a** | $\frac{1}{25}$ | **b** | $\frac{1}{64}$ | **c** | $\frac{1}{32}$ | **d** | $\frac{1}{36}$ |
| **5** | **a** | $\frac{3x^3}{2}$ | **b** | $5x^2$ | | | | |
| | **c** | $3x$ | **d** | $\frac{y}{2x^2}$ | | | | |
| | **e** | $y^{\frac{1}{2}}$ | **f** | $c^{-3}$ | | | | |
| | **g** | $2x^6$ | **h** | $x$ | | | | |
| **6** | **a** | $\frac{1}{2}$ | **b** | $\frac{1}{9}$ | **c** | $\frac{8}{3}$ | | |
| | **d** | $\frac{1}{4}$ | **e** | $\frac{4}{3}$ | **f** | $\frac{16}{9}$ | | |
| **7** | **a** | $x^{-1}$ | **b** | $x^{-7}$ | **c** | $x^{\frac{1}{4}}$ | | |
| | **d** | $x^{\frac{2}{5}}$ | **e** | $x^{-\frac{1}{3}}$ | **f** | $x^{-\frac{2}{3}}$ | | |
| **8** | **a** | $\frac{1}{x^3}$ | **b** | 1 | **c** | $\sqrt[5]{x}$ | | |
| | **d** | $\sqrt[5]{x^2}$ | **e** | $\frac{1}{\sqrt{x}}$ | **f** | $\frac{1}{\sqrt[4]{x^3}}$ | | |
| **9** | **a** | $5x^{\frac{1}{2}}$ | **b** | $2x^{-3}$ | **c** | $\frac{1}{3}x^{-4}$ | | |
| | **d** | $2x^{-\frac{1}{2}}$ | **e** | $4x^{-\frac{1}{3}}$ | **f** | $3x^0$ | | |
| **10** | **a** | $x^3 + x^{-2}$ | **b** | $x^3 + x$ | **c** | $x^{-2} + x^{-7}$ | | |

Pearson

# Solving linear simultaneous equations using the elimination method

**A LEVEL LINKS**
**Scheme of work:** 1c. Equations – quadratic/linear simultaneous

## Key points

- Two equations are simultaneous when they are both true at the same time.
- Solving simultaneous linear equations in two unknowns involves finding the value of each unknown which works for both equations.
- Make sure that the coefficient of one of the unknowns is the same in both equations.
- Eliminate this equal unknown by either subtracting or adding the two equations.

## Examples

**Example 1** Solve the simultaneous equations $3x + y = 5$ and $x + y = 1$

| | |
|---|---|
| $$\begin{array}{rl} & 3x + y = 5 \\ - & \;\;x + y = 1 \\ \hline & 2x \quad\;\; = 4 \end{array}$$ So $x = 2$ | **1** Subtract the second equation from the first equation to eliminate the $y$ term. |
| Using $x + y = 1$<br>$2 + y = 1$<br>So $y = -1$ | **2** To find the value of $y$, substitute $x = 2$ into one of the original equations. |
| Check:<br>equation 1: $3 \times 2 + (-1) = 5$ YES<br>equation 2: $2 + (-1) = 1$ YES | **3** Substitute the values of $x$ and $y$ into both equations to check your answers. |

**Example 2** Solve $x + 2y = 13$ and $5x - 2y = 5$ simultaneously.

| | |
|---|---|
| $$\begin{array}{rl} & x + 2y = 13 \\ + & 5x - 2y = \;\;5 \\ \hline & 6x \qquad = 18 \end{array}$$ So $x = 3$ | **1** Add the two equations together to eliminate the $y$ term. |
| Using $x + 2y = 13$<br>$3 + 2y = 13$<br>So $y = 5$ | **2** To find the value of $y$, substitute $x = 3$ into one of the original equations. |
| Check:<br>equation 1: $3 + 2 \times 5 = 13$ YES<br>equation 2: $5 \times 3 - 2 \times 5 = 5$ YES | **3** Substitute the values of $x$ and $y$ into both equations to check your answers. |

**Example 3** Solve $2x + 3y = 2$ and $5x + 4y = 12$ simultaneously.

| | |
|---|---|
| $(2x + 3y = 2) \times 4 \rightarrow \quad 8x + 12y = 8$<br>$(5x + 4y = 12) \times 3 \rightarrow \quad 15x + 12y = 36$<br>$7x \quad = 28$<br><br>So $x = 4$ | **1** Multiply the first equation by 4 and the second equation by 3 to make the coefficient of $y$ the same for both equations. Then subtract the first equation from the second equation to eliminate the $y$ term. |
| Using $2x + 3y = 2$<br>$2 \times 4 + 3y = 2$<br>So $y = -2$ | **2** To find the value of $y$, substitute $x = 4$ into one of the original equations. |
| Check:<br>equation 1: $2 \times 4 + 3 \times (-2) = 2$ YES<br>equation 2: $5 \times 4 + 4 \times (-2) = 12$ YES | **3** Substitute the values of $x$ and $y$ into both equations to check your answers. |

# Practice

Solve these simultaneous equations.

**1** $4x + y = 8$
$x + y = 5$

**2** $3x + y = 7$
$3x + 2y = 5$

**3** $4x + y = 3$
$3x - y = 11$

**4** $3x + 4y = 7$
$x - 4y = 5$

**5** $2x + y = 11$
$x - 3y = 9$

**6** $2x + 3y = 11$
$3x + 2y = 4$

# Solving linear simultaneous equations using the substitution method

**A LEVEL LINKS**
**Scheme of work:** 1c. Equations – quadratic/linear simultaneous
**Textbook:** Pure Year 1, 3.1 Linear simultaneous equations

## Key points

- The subsitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

## Examples

**Example 4** Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

| | |
|---|---|
| $5x + 3(2x + 1) = 14$ | **1** Substitute $2x + 1$ for $y$ into the second equation. |
| $5x + 6x + 3 = 14$<br>$11x + 3 = 14$ | **2** Expand the brackets and simplify. |
| $11x = 11$<br>So $x = 1$ | **3** Work out the value of $x$. |
| Using $y = 2x + 1$<br>$y = 2 \times 1 + 1$<br>So $y = 3$ | **4** To find the value of $y$, substitute $x = 1$ into one of the original equations. |
| Check:<br>equation 1: $3 = 2 \times 1 + 1$ YES<br>equation 2: $5 \times 1 + 3 \times 3 = 14$ YES | **5** Substitute the values of $x$ and $y$ into both equations to check your answers. |

**Example 5** Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

| | |
|---|---|
| $y = 2x - 16$ | **1** Rearrange the first equation. |
| $4x + 3(2x - 16) = -3$ | **2** Substitute $2x - 16$ for $y$ into the second equation. |
| $4x + 6x - 48 = -3$<br>$10x - 48 = -3$ | **3** Expand the brackets and simplify. |
| $10x = 45$<br>So $x = 4\frac{1}{2}$ | **4** Work out the value of $x$. |
| Using $y = 2x - 16$<br>$y = 2 \times 4\frac{1}{2} - 16$<br>So $y = -7$ | **5** To find the value of $y$, substitute $x = 4\frac{1}{2}$ into one of the original equations. |
| Check:<br>equation 1: $2 \times 4\frac{1}{2} - (-7) = 16$ YES<br>equation 2: $4 \times 4\frac{1}{2} + 3 \times (-7) = -3$ YES | **6** Substitute the values of $x$ and $y$ into both equations to check your answers. |

## Practice

Solve these simultaneous equations.

**7** $y = x - 4$
$2x + 5y = 43$

**8** $y = 2x - 3$
$5x - 3y = 11$

**9** $2y = 4x + 5$
$9x + 5y = 22$

**10** $2x = y - 2$
$8x - 5y = -11$

**11** $3x + 4y = 8$
$2x - y = -13$

**12** $3y = 4x - 7$
$2y = 3x - 4$

**13** $3x = y - 1$
$2y - 2x = 3$

**14** $3x + 2y + 1 = 0$
$4y = 8 - x$

## Extend

**15** Solve the simultaneous equations $3x + 5y - 20 = 0$ and $2(x + y) = \frac{3(y - x)}{4}$.

# Answers

**1** $x = 1, y = 4$

**2** $x = 3, y = -2$

**3** $x = 2, y = -5$

**4** $x = 3, y = -\frac{1}{2}$

**5** $x = 6, y = -1$

**6** $x = -2, y = 5$

**7** $x = 9, y = 5$

**8** $x = -2, y = -7$

**9** $x = \frac{1}{2}, y = 3\frac{1}{2}$

**10** $x = \frac{1}{2}, y = 3$

**11** $x = -4, y = 5$

**12** $x = -2, y = -5$

**13** $x = \frac{1}{4}, y = 1\frac{3}{4}$

**14** $x = -2, y = 2\frac{1}{2}$

**15** $x = -2\frac{1}{2}, y = 5\frac{1}{2}$

edexcel

# Solving linear and quadratic simultaneous equations

**A LEVEL LINKS**
**Scheme of work:** 1c. Equations – quadratic/linear simultaneous

## Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

## Examples

**Example 1** Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

| | |
|---|---|
| $x^2 + (x + 1)^2 = 13$ | **1** Substitute $x + 1$ for $y$ into the second equation. |
| $x^2 + x^2 + x + x + 1 = 13$<br>$2x^2 + 2x + 1 = 13$ | **2** Expand the brackets and simplify. |
| $2x^2 + 2x - 12 = 0$<br>$(2x - 4)(x + 3) = 0$ | **3** Factorise the quadratic equation. |
| So $x = 2$ or $x = -3$ | **4** Work out the values of $x$. |
| Using $y = x + 1$<br>When $x = 2$, $y = 2 + 1 = 3$<br>When $x = -3$, $y = -3 + 1 = -2$ | **5** To find the value of $y$, substitute both values of $x$ into one of the original equations. |
| So the solutions are<br>$x = 2,\ y = 3$ and $x = -3,\ y = -2$ | |
| Check:<br>equation 1: $3 = 2 + 1$ YES<br>and $-2 = -3 + 1$ YES<br>equation 2: $2^2 + 3^2 = 13$ YES<br>and $(-3)^2 + (-2)^2 = 13$ YES | **6** Substitute both pairs of values of $x$ and $y$ into both equations to check your answers. |

Pearson

**Example 2** Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

| | |
|---|---|
| $x = \dfrac{5-3y}{2}$ | **1** Rearrange the first equation. |
| $2y^2 + \left(\dfrac{5-3y}{2}\right)y = 12$ | **2** Substitute $\dfrac{5-3y}{2}$ for $x$ into the second equation. Notice how it is easier to substitute for $x$ than for $y$. |
| $2y^2 + \dfrac{5y-3y^2}{2} = 12$<br>$4y^2 + 5y - 3y^2 = 24$<br>$y^2 + 5y - 24 = 0$ | **3** Expand the brackets and simplify. |
| $(y + 8)(y - 3) = 0$ | **4** Factorise the quadratic equation. |
| So $y = -8$ or $y = 3$ | **5** Work out the values of $y$. |
| Using $2x + 3y = 5$<br>When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$<br>When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$ | **6** To find the value of $x$, substitute both values of $y$ into one of the original equations. |
| So the solutions are<br>$x = 14.5$, $y = -8$ and $x = -2$, $y = 3$ | |
| Check:<br>equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES<br>and $2 \times (-2) + 3 \times 3 = 5$ YES<br>equation 2: $2\times(-8)^2 + 14.5\times(-8) = 12$ YES<br>and $2 \times (3)^2 + (-2) \times 3 = 12$ YES | **7** Substitute both pairs of values of $x$ and $y$ into both equations to check your answers. |

## Practice

Solve these simultaneous equations.

**1** $y = 2x + 1$
$x^2 + y^2 = 10$

**2** $y = 6 - x$
$x^2 + y^2 = 20$

**3** $y = x - 3$
$x^2 + y^2 = 5$

**4** $y = 9 - 2x$
$x^2 + y^2 = 17$

**5** $y = 3x - 5$
$y = x^2 - 2x + 1$

**6** $y = x - 5$
$y = x^2 - 5x - 12$

**7** $y = x + 5$
$x^2 + y^2 = 25$

**8** $y = 2x - 1$
$x^2 + xy = 24$

**9** $y = 2x$
$y^2 - xy = 8$

**10** $2x + y = 11$
$xy = 15$

## Extend

**11** $x - y = 1$
$x^2 + y^2 = 3$

**12** $y - x = 2$
$x^2 + xy = 3$

## Answers

**1** $x = 1, y = 3$

$x = -\frac{9}{5},\ y = -\frac{13}{5}$

**2** $x = 2, y = 4$

$x = 4, y = 2$

**3** $x = 1, y = -2$

$x = 2, y = -1$

**4** $x = 4, y = 1$

$x = \frac{16}{5},\ y = \frac{13}{5}$

**5** $x = 3, y = 4$

$x = 2, y = 1$

**6** $x = 7, y = 2$

$x = -1, y = -6$

**7** $x = 0, y = 5$

$x = -5, y = 0$

**8** $x = -\frac{8}{3}, y = -\frac{19}{3}$

$x = 3, y = 5$

**9** $x = -2, y = -4$

$x = 2, y = 4$

**10** $x = \frac{5}{2}, y = 6$

$x = 3, y = 5$

**11** $x = \frac{1+\sqrt{5}}{2}, y = \frac{-1+\sqrt{5}}{2}$

$x = \frac{1-\sqrt{5}}{2}, y = \frac{-1-\sqrt{5}}{2}$

**12** $x = \frac{-1+\sqrt{7}}{2}, y = \frac{3+\sqrt{7}}{2}$

$x = \frac{-1-\sqrt{7}}{2}, y = \frac{3-\sqrt{7}}{2}$

# Solving simultaneous equations graphically

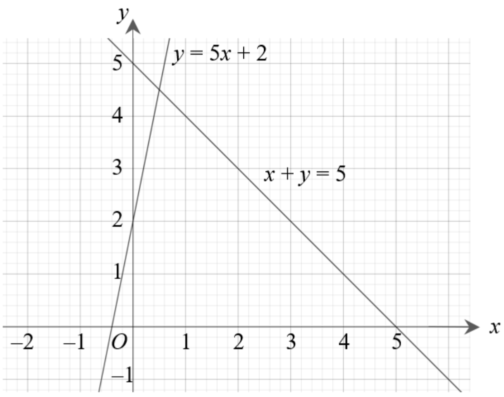
**A LEVEL LINKS**
**Scheme of work:** 1c. Equations – quadratic/linear simultaneous

## Key points

- You can solve any pair of simultaneous equations by drawing the graph of both equations and finding the point/points of intersection.

## Examples

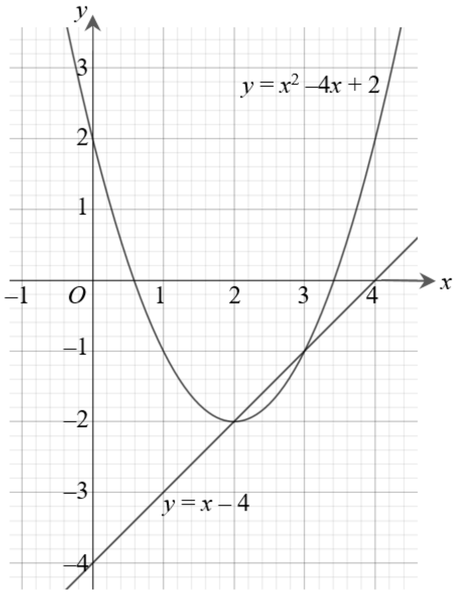
**Example 1** Solve the simultaneous equations $y = 5x + 2$ and $x + y = 5$ graphically.

| | |
|---|---|
| $y = 5 - x$ | **1** Rearrange the equation $x + y = 5$ to make $y$ the subject. |
| $y = 5 - x$ has gradient $-1$ and $y$-intercept 5.<br>$y = 5x + 2$ has gradient 5 and $y$-intercept 2.<br> | **2** Plot both graphs on the same grid using the gradients and $y$-intercepts. |
| Lines intersect at<br>$x = 0.5$, $y = 4.5$ | **3** The solutions of the simultaneous equations are the point of intersection. |
| Check:<br>First equation $y = 5x + 2$:<br>$4.5 = 5 \times 0.5 + 2$ YES<br>Second equation $x + y = 5$:<br>$0.5 + 4.5 = 5$ YES | **4** Check your solutions by substituting the values into both equations. |

Example 2 Solve the simultaneous equations $y = x - 4$ and $y = x^2 - 4x + 2$ graphically.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 2 | –1 | –2 | –1 | 2 |

1 Construct a table of values and calculate the points for the quadratic equation.




2 Plot the graph.

3 Plot the linear graph on the same grid using the gradient and $y$-intercept.
$y = x – 4$ has gradient 1 and $y$-intercept –4.

The line and curve intersect at
$x = 3, y = −1$ and $x = 2, y = −2$

4 The solutions of the simultaneous equations are the points of intersection.

Check:
First equation $y = x − 4$:
$−1 = 3 − 4$ YES
$−2 = 2 − 4$ YES
Second equation $y = x^2 − 4x + 2$:
$−1 = 3^2 − 4 × 3 + 2$ YES
$−2 = 2^2 − 4 × 2 + 2$ YES

5 Check your solutions by substituting the values into both equations.

## Practice

1 Solve these pairs of simultaneous equations graphically.

a $y = 3x − 1$ and $y = x + 3$

b $y = x − 5$ and $y = 7 − 5x$

c $y = 3x + 4$ and $y = 2 − x$

2 Solve these pairs of simultaneous equations graphically.

a $x + y = 0$ and $y = 2x + 6$

b $4x + 2y = 3$ and $y = 3x − 1$

c $2x + y + 4 = 0$ and $2y = 3x − 1$

**Hint**

Rearrange the equation to make $y$ the subject.

**3** Solve these pairs of simultaneous equations graphically.

**a** $y = x - 1$ and $y = x^2 - 4x + 3$

**b** $y = 1 - 3x$ and $y = x^2 - 3x - 3$

**c** $y = 3 - x$ and $y = x^2 + 2x + 5$

**4** Solve the simultaneous equations $x + y = 1$ and $x^2 + y^2 = 25$ graphically.

## Extend

**5** **a** Solve the simultaneous equations $2x + y = 3$ and $x^2 + y = 4$

**i** graphically

**ii** algebraically to 2 decimal places.

**b** Which method gives the more accurate solutions? Explain your answer.

## Answers

**1** **a** $x = 2, y = 5$
**b** $x = 2, y = -3$
**c** $x = -0.5, y = 2.5$

**2** **a** $x = -2, y = 2$
**b** $x = 0.5, y = 0.5$
**c** $x = -1, y = -2$

**3** **a** $x = 1, y = 0$ and $x = 4, y = 3$
**b** $x = -2, y = 7$ and $x = 2, y = -5$
**c** $x = -2, y = 5$ and $x = -1, y = 4$

**4** $x = -3, y = 4$ and $x = 4, y = -3$

**5** **a** **i** $x = 2.5, y = -2$ and $x = -0.5, y = 4$
**ii** $x = 2.41, y = -1.83$ and $x = -0.41, y = 3.83$
**b** Solving algebraically gives the more accurate solutions as the solutions from the graph are only estimates, based on the accuracy of your graph.

# Linear inequalities

**A LEVEL LINKS**
**Scheme of work:** 1d. Inequalities – linear and quadratic (including graphical solutions)

## Key points

- Solving linear inequalities uses similar methods to those for solving linear equations.
- When you multiply or divide an inequality by a negative number you need to reverse the inequality sign, e.g. < becomes >.

## Examples

**Example 1** Solve $-8 \leq 4x < 16$

| | |
|---|---|
| $-8 \leq 4x < 16$<br>$-2 \leq x < 4$ | Divide all three terms by 4. |

**Example 2** Solve $4 \leq 5x < 10$

| | |
|---|---|
| $4 \leq 5x < 10$<br>$\frac{4}{5} \leq x < 2$ | Divide all three terms by 5. |

**Example 3** Solve $2x - 5 < 7$

| | |
|---|---|
| $2x - 5 < 7$<br>$2x < 12$<br>$x < 6$ | **1** Add 5 to both sides.<br>**2** Divide both sides by 2. |

**Example 4** Solve $2 - 5x \geq -8$

| | |
|---|---|
| $2 - 5x \geq -8$<br>$-5x \geq -10$<br>$x \leq 2$ | **1** Subtract 2 from both sides.<br>**2** Divide both sides by −5. Remember to reverse the inequality when dividing by a negative number. |

**Example 5** Solve $4(x - 2) > 3(9 - x)$

| | |
|---|---|
| $4(x - 2) > 3(9 - x)$<br>$4x - 8 > 27 - 3x$<br>$7x - 8 > 27$<br>$7x > 35$<br>$x > 5$ | **1** Expand the brackets.<br>**2** Add $3x$ to both sides.<br>**3** Add 8 to both sides.<br>**4** Divide both sides by 7. |

## Practice

**1** Solve these inequalities.

**a** $4x > 16$ **b** $5x - 7 \leq 3$ **c** $1 \geq 3x + 4$

**d** $5 - 2x < 12$ **e** $\frac{x}{2} \geq 5$ **f** $8 < 3 - \frac{x}{3}$

**2** Solve these inequalities.

**a** $\frac{x}{5} < -4$ **b** $10 \geq 2x + 3$ **c** $7 - 3x > -5$

**3** Solve

**a** $2 - 4x \geq 18$ **b** $3 \leq 7x + 10 < 45$ **c** $6 - 2x \geq 4$

**d** $4x + 17 < 2 - x$ **e** $4 - 5x < -3x$ **f** $-4x \geq 24$

**4** Solve these inequalities.

**a** $3t + 1 < t + 6$ **b** $2(3n - 1) \geq n + 5$

**5** Solve.

**a** $3(2 - x) > 2(4 - x) + 4$ **b** $5(4 - x) > 3(5 - x) + 2$

## Extend

**6** Find the set of values of $x$ for which $2x + 1 > 11$ and $4x - 2 > 16 - 2x$.

## Answers

**1** **a** $x > 4$ **b** $x \leq 2$ **c** $x \leq -1$

**d** $x > -\frac{7}{2}$ **e** $x \geq 10$ **f** $x < -15$

**2** **a** $x < -20$ **b** $x \leq 3.5$ **c** $x < 4$

**3** **a** $x \leq -4$ **b** $-1 \leq x < 5$ **c** $x \leq 1$

**d** $x < -3$ **e** $x > 2$ **f** $x \leq -6$

**4** **a** $t < \frac{5}{2}$ **b** $n \geq \frac{7}{5}$

**5** **a** $x < -6$ **b** $x < \frac{3}{2}$

**6** $x > 5$ (which also satisfies $x > 3$)

# Quadratic inequalities

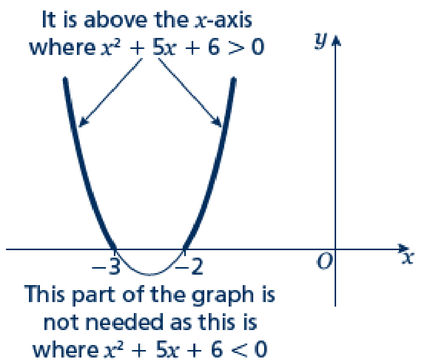
**A LEVEL LINKS**
**Scheme of work:** 1d. Inequalities – linear and quadratic (including graphical solutions)

## Key points

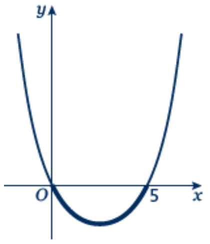
- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

## Examples

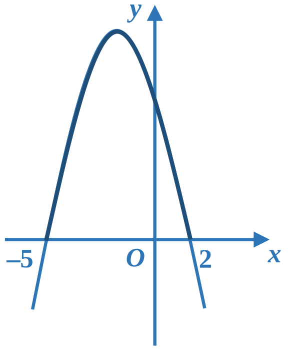
**Example 1** Find the set of values of $x$ which satisfy $x^2 + 5x + 6 > 0$

| | |
|---|---|
| $x^2 + 5x + 6 = 0$<br>$(x + 3)(x + 2) = 0$<br>$x = -3$ or $x = -2$ | **1** Solve the quadratic equation by factorising. |
| <br> | **2** Sketch the graph of $y = (x + 3)(x + 2)$<br>**3** Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$ |
| $x < -3$ or $x > -2$ | **4** Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$ |

**Example 2** Find the set of values of $x$ which satisfy $x^2 - 5x \leq 0$

| | |
|---|---|
| $x^2 - 5x = 0$<br>$x(x - 5) = 0$<br>$x = 0$ or $x = 5$ | **1** Solve the quadratic equation by factorising. |
| <br> | **2** Sketch the graph of $y = x(x - 5)$<br>**3** Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$ |
| $0 \leq x \leq 5$ | **4** Write down the values which satisfy the inequality $x^2 - 5x \leq 0$ |

**Example 3** Find the set of values of $x$ which satisfy $-x^2 - 3x + 10 \geq 0$

| | |
|---|---|
| $-x^2 - 3x + 10 = 0$<br>$(-x + 2)(x + 5) = 0$<br>$x = 2$ or $x = -5$ | **1** Solve the quadratic equation by factorising. |
|  | **2** Sketch the graph of $y = (-x + 2)(x + 5) = 0$<br><br>**3** Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$ |
| $-5 \leq x \leq 2$ | **3** Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$ |

## Practice

**1** Find the set of values of $x$ for which $(x + 7)(x - 4) \leq 0$

**2** Find the set of values of $x$ for which $x^2 - 4x - 12 \geq 0$

**3** Find the set of values of $x$ for which $2x^2 - 7x + 3 < 0$

**4** Find the set of values of $x$ for which $4x^2 + 4x - 3 > 0$

**5** Find the set of values of $x$ for which $12 + x - x^2 \geq 0$

## Extend

Find the set of values which satisfy the following inequalities.

**6** $x^2 + x \leq 6$

**7** $x(2x - 9) < -10$

**8** $6x^2 \geq 15 + x$

## Answers

**1** $-7 \le x \le 4$

**2** $x \le -2$ or $x \ge 6$

**3** $\frac{1}{2} < x < 3$

**4** $x < -\frac{3}{2}$ or $x > \frac{1}{2}$

**5** $-3 \le x \le 4$

**6** $-3 \le x \le 2$

**7** $2 < x < 2\frac{1}{2}$

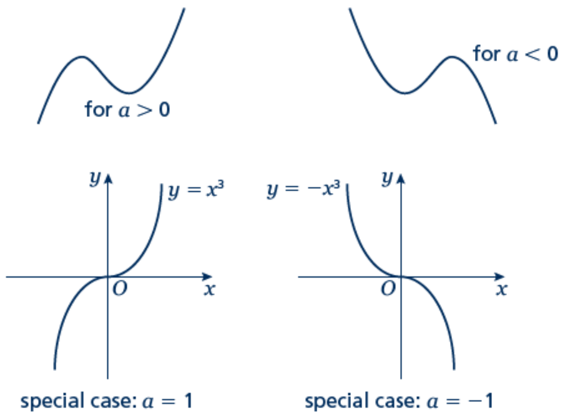
**8** $x \le -\frac{3}{2}$ or $x \ge \frac{5}{3}$

# Sketching cubic and reciprocal graphs

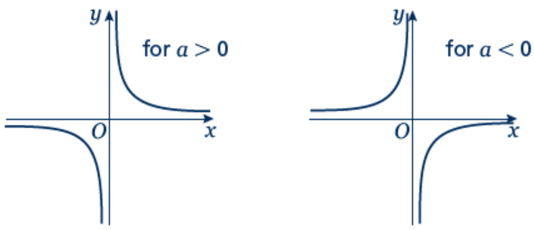
**A LEVEL LINKS**
**Scheme of work:** 1e. Graphs – cubic, quartic and reciprocal

## Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.



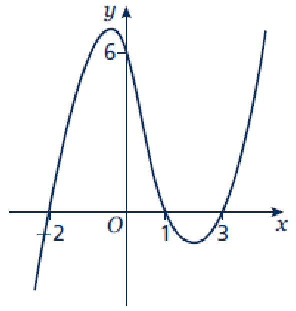
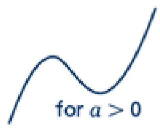

- The graph of a reciprocal function of the form $y = \frac{a}{x}$ has one of the shapes shown here.



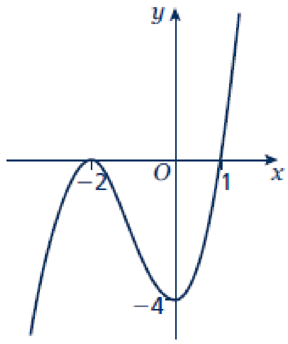
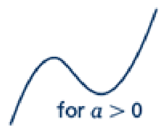

- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the $y$-axis substitute $x = 0$ into the function.
- To find where the curve intersects the $x$-axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \frac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x - 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

# Examples

**Example 1** Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

| To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape. | |
|---|---|
| When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$<br>$= (-3) \times (-1) \times 2 = 6$<br>The graph intersects the $y$-axis at $(0, 6)$<br><br>When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$<br>So $x = 3$, $x = 1$ or $x = -2$<br>The graph intersects the $x$-axis at<br>$(-2, 0)$, $(1, 0)$ and $(3, 0)$ | **1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, $(x, y)$.<br>**2** Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$ |
|  | **3** Sketch the graph.<br>$a = 1 > 0$ so the graph has the shape:<br> |

**Example 2** Sketch the graph of $y = (x + 2)^2(x - 1)$

| To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape. | |
|---|---|
| When $x = 0$, $y = (0 + 2)^2(0 - 1)$<br>$= 2^2 \times (-1) = -4$<br>The graph intersects the $y$-axis at $(0, -4)$ | **1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. |
| When $y = 0$, $(x + 2)^2(x - 1) = 0$<br>So $x = -2$ or $x = 1$<br><br>$(-2, 0)$ is a turning point as $x = -2$ is a double root.<br>The graph crosses the $x$-axis at $(1, 0)$ | **2** Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$ |
|  | **3** $a = 1 > 0$ so the graph has the shape:<br> |

## Practice

**1** Here are six equations.

**A** $y = \frac{5}{x}$  **B** $y = x^2 + 3x - 10$  **C** $y = x^3 + 3x^2$

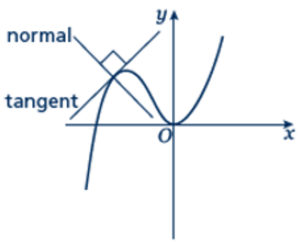
**D** $y = 1 - 3x^2 - x^3$  **E** $y = x^3 - 3x^2 - 1$  **F** $x + y = 5$

**Hint**

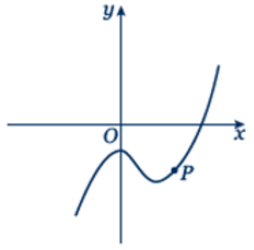
Find where each of the cubic equations cross the $y$-axis.

Here are six graphs.

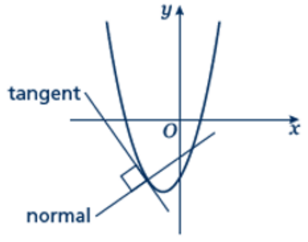
**i**



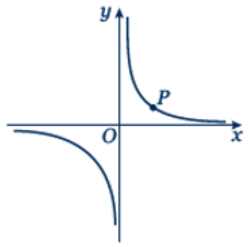

**ii**



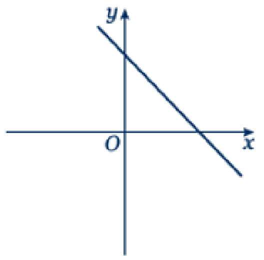

**iii**



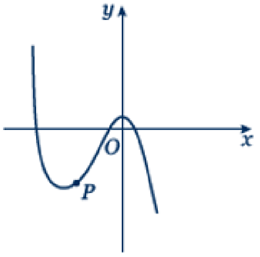

**iv**




**v**




**vi**




**a** Match each graph to its equation.

**b** Copy the graphs ii, iv and vi and draw the tangent and normal each at point $P$.

Sketch the following graphs

**2** $y = 2x^3$

**3** $y = x(x - 2)(x + 2)$

**4** $y = (x + 1)(x + 4)(x - 3)$

**5** $y = (x + 1)(x - 2)(1 - x)$

**6** $y = (x - 3)^2(x + 1)$

**7** $y = (x - 1)^2(x - 2)$

**8** $y = \frac{3}{x}$

**Hint:** Look at the shape of $y = \frac{a}{x}$ in the second key point.

**9** $y = -\frac{2}{x}$

## Extend

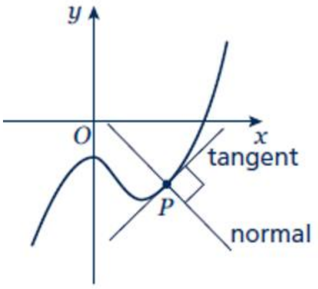
**10** Sketch the graph of $y = \frac{1}{x+2}$

**11** Sketch the graph of $y = \frac{1}{x-1}$

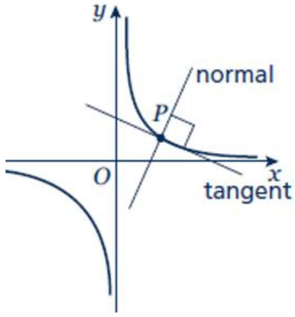
# Answers

**1** **a** i – C
ii – E
iii – B
iv – A
v – F
vi – D

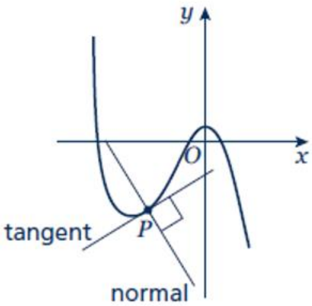
**b** **ii**



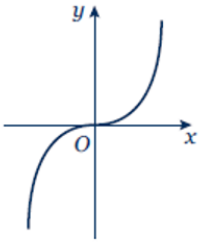

**iv**



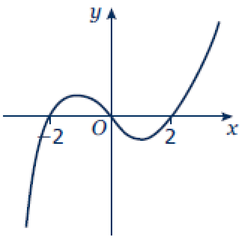

**vi**



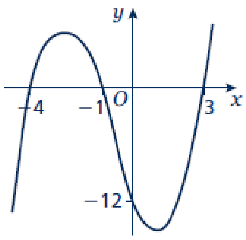

**2**



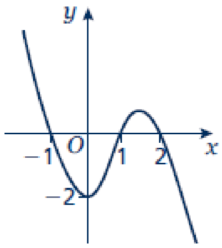

**3**



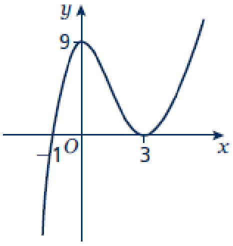

**4**



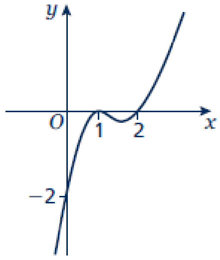

**5**



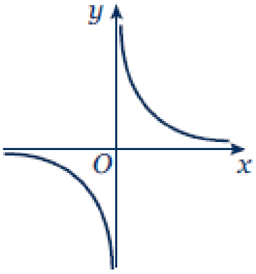
6



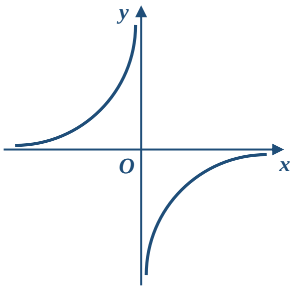

7



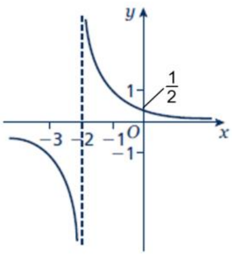

8



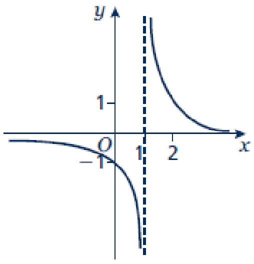

9




10




11



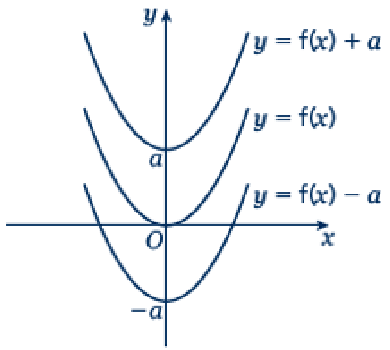
# Translating graphs

**A LEVEL LINKS**
**Scheme of work:** 1f. Transformations – transforming graphs – f($x$) notation

## Key points

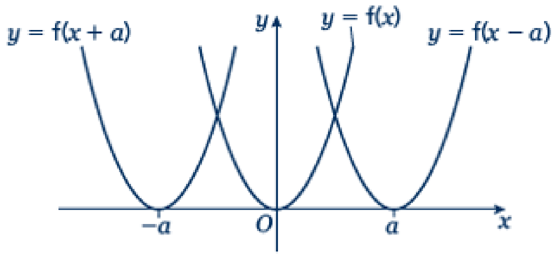
- The transformation $y = \mathrm{f}(x) \pm a$ is a translation of $y = \mathrm{f}(x)$ parallel to the $y$-axis; it is a vertical translation.

  As shown on the graph,
  - $y = \mathrm{f}(x) + a$ translates $y = \mathrm{f}(x)$ up
  - $y = \mathrm{f}(x) - a$ translates $y = \mathrm{f}(x)$ down.




- The transformation $y = \mathrm{f}(x \pm a)$ is a translation of $y = \mathrm{f}(x)$ parallel to the $x$-axis; it is a horizontal translation.

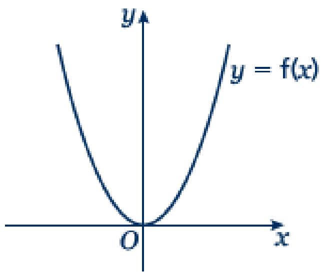
  As shown on the graph,
  - $y = \mathrm{f}(x + a)$ translates $y = \mathrm{f}(x)$ to the left
  - $y = \mathrm{f}(x - a)$ translates $y = \mathrm{f}(x)$ to the right.

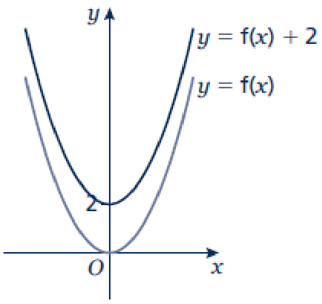



## Examples

**Example 1** The graph shows the function $y = \mathrm{f}(x)$.

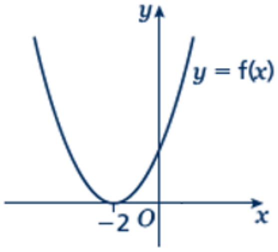
Sketch the graph of $y = \mathrm{f}(x) + 2$.

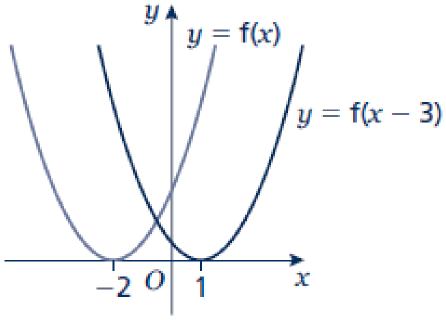



| | |
|---|---|
|  | For the function $y = \mathrm{f}(x) + 2$ translate the function $y = \mathrm{f}(x)$ 2 units up. |

**Example 2** The graph shows the function $y = \mathrm{f}(x)$.

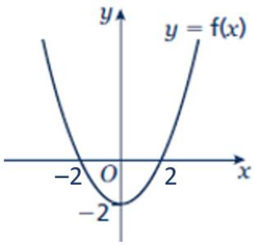
Sketch the graph of $y = \mathrm{f}(x - 3)$.



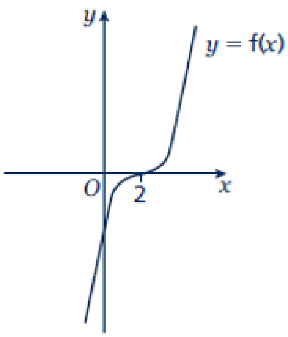

| | |
|---|---|
|   | For the function $y = \mathrm{f}(x - 3)$ translate the function $y = \mathrm{f}(x)$ 3 units right. |

## Practice

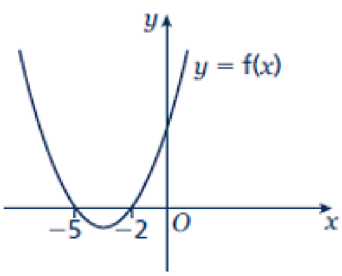
**1** The graph shows the function $y = \mathrm{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \mathrm{f}(x) + 4$ and $y = \mathrm{f}(x + 2)$.



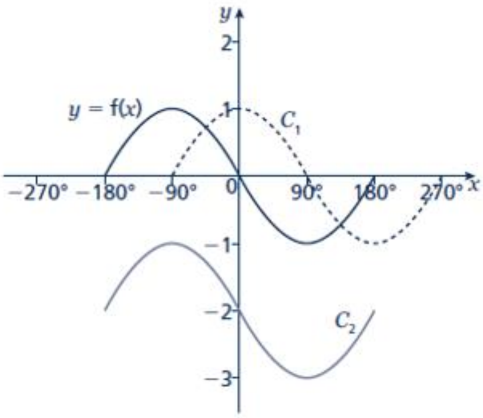

**2** The graph shows the function $y = \mathrm{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \mathrm{f}(x + 3)$ and $y = \mathrm{f}(x) - 3$.



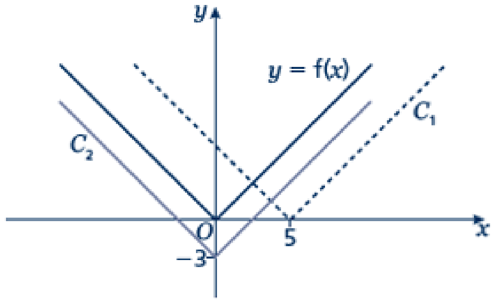

**3** The graph shows the function $y = \mathrm{f}(x)$.
Copy the graph and on the same axes sketch the graph of $y = \mathrm{f}(x - 5)$.



4 The graph shows the function $y = \mathrm{f}(x)$ and two transformations of $y = \mathrm{f}(x)$, labelled $C_1$ and $C_2$. Write down the equations of the translated curves $C_1$ and $C_2$ in function form.



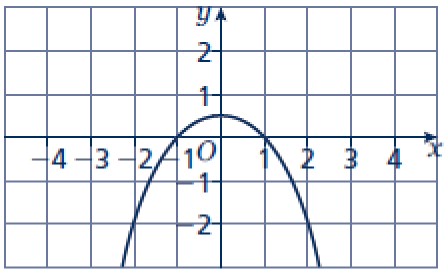

5 The graph shows the function $y = \mathrm{f}(x)$ and two transformations of $y = \mathrm{f}(x)$, labelled $C_1$ and $C_2$. Write down the equations of the translated curves $C_1$ and $C_2$ in function form.




6 The graph shows the function $y = \mathrm{f}(x)$.

**a** Sketch the graph of $y = \mathrm{f}(x) + 2$

**b** Sketch the graph of $y = \mathrm{f}(x + 2)$



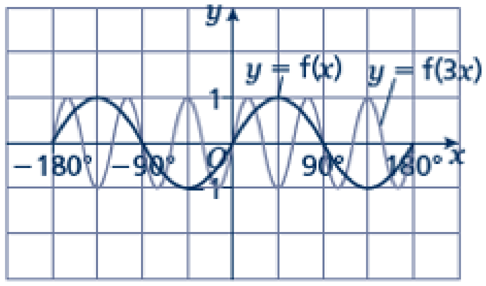
edexcel

# Stretching graphs

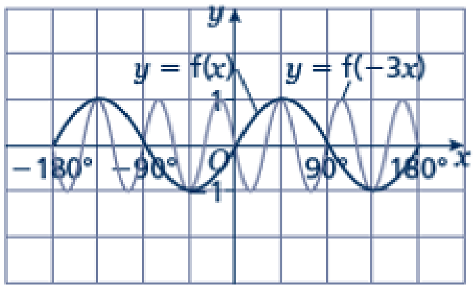
**A LEVEL LINKS**
**Scheme of work:** 1f. Transformations – transforming graphs – f($x$) notation
**Textbook:** Pure Year 1, 4.6 Stretching graphs

## Key points

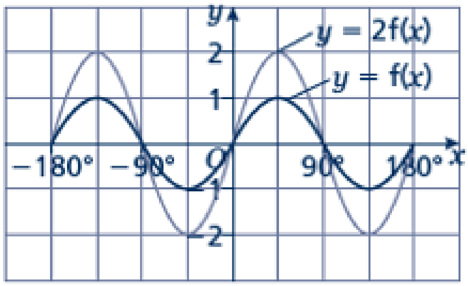
- The transformation $y = \text{f}(ax)$ is a horizontal stretch of $y = \text{f}(x)$ with scale factor $\frac{1}{a}$ parallel to the $x$-axis.



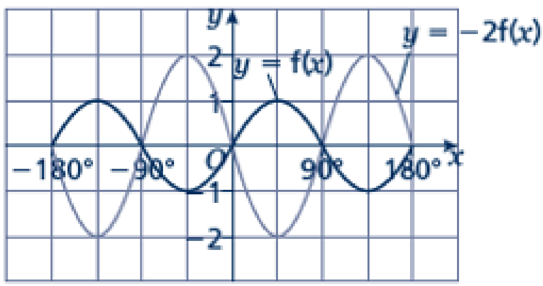

- The transformation $y = \text{f}(-ax)$ is a horizontal stretch of $y = \text{f}(x)$ with scale factor $\frac{1}{a}$ parallel to the $x$-axis and then a reflection in the $y$-axis.




- The transformation $y = a\text{f}(x)$ is a vertical stretch of $y = \text{f}(x)$ with scale factor $a$ parallel to the $y$-axis.




- The transformation $y = -a\text{f}(x)$ is a vertical stretch of $y = \text{f}(x)$ with scale factor $a$ parallel to the $y$-axis and then a reflection in the $x$-axis.

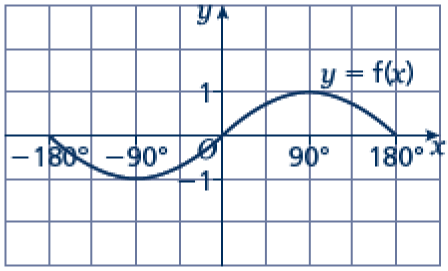


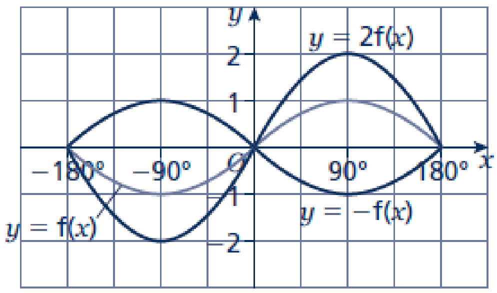

Pearson

# Examples

**Example 3** The graph shows the function $y = f(x)$.

Sketch and label the graphs of $y = 2f(x)$ and $y = –f(x)$.



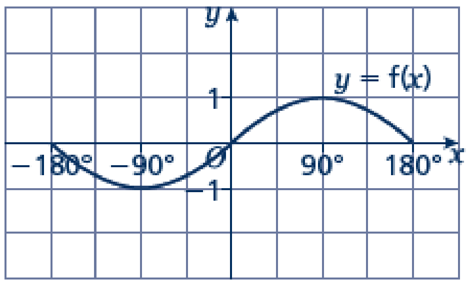



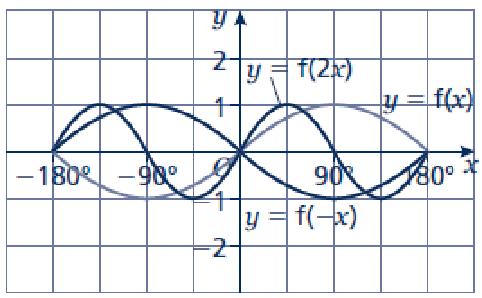

The function $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2 parallel to the $y$-axis.

The function $y = −f(x)$ is a reflection of $y = f(x)$ in the $x$-axis.

**Example 4** The graph shows the function $y = f(x)$.

Sketch and label the graphs of $y = f(2x)$ and $y = f(–x)$.




The function $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$ parallel to the $x$-axis.

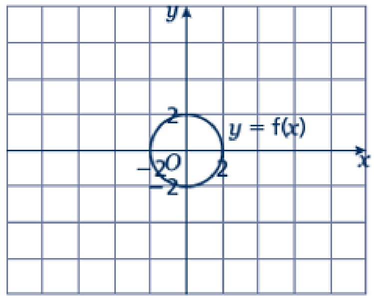
The function $y = f(−x)$ is a reflection of $y = f(x)$ in the $y$-axis.

## Practice

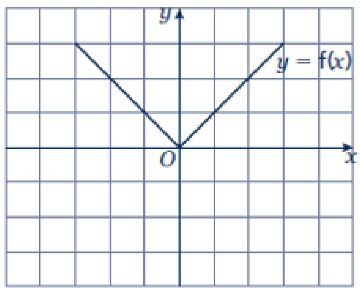
**7** The graph shows the function $y = f(x)$.

**a** Copy the graph and on the same axes sketch and label the graph of $y = 3f(x)$.

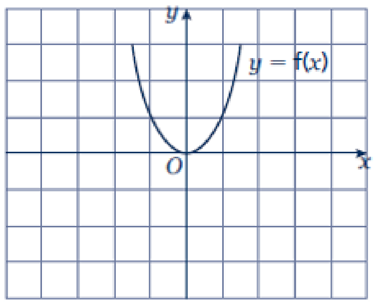
**b** Make another copy of the graph and on the same axes sketch and label the graph of $y = f(2x)$.



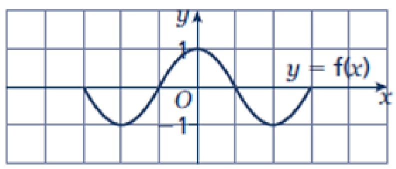

**8** The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and label the graphs of
$y = –2f(x)$ and $y = f(3x)$.



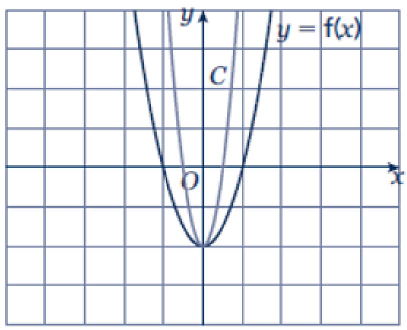

**9** The graph shows the function $y = f(x)$.
Copy the graph and, on the same axes, sketch and label the graphs of
$y = –f(x)$ and $y = f\left(\frac{1}{2}x\right)$.



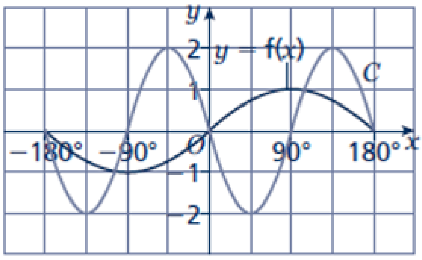

**10** The graph shows the function $y = f(x)$.
Copy the graph and, on the same axes, sketch the graph of $y = –f(2x)$.




**11** The graph shows the function $y = f(x)$ and a transformation, labelled $C$.
Write down the equation of the translated curve $C$ in function form.



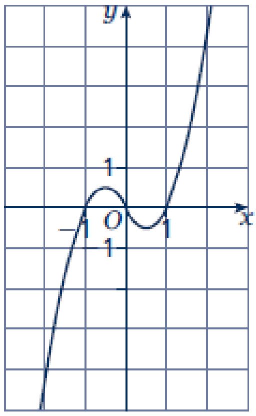
**12** The graph shows the function $y = \mathrm{f}(x)$ and a transformation labelled $C$.
Write down the equation of the translated curve $C$ in function form.




**13** The graph shows the function $y = \mathrm{f}(x)$.

**a** Sketch the graph of $y = -\mathrm{f}(x)$.

**b** Sketch the graph of $y = 2\mathrm{f}(x)$.




## Extend

**14** **a** Sketch and label the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = (x - 1)(x + 1)$.

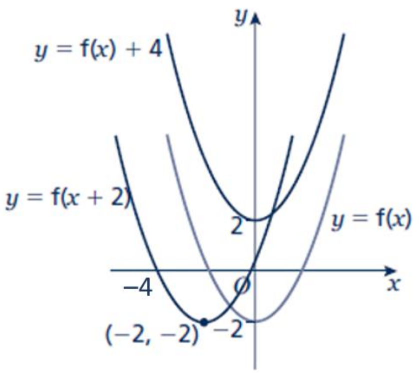
**b** On the same axes, sketch and label the graphs of $y = \mathrm{f}(x) - 2$ and $y = \mathrm{f}(x + 2)$.

**15** **a** Sketch and label the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = -(x + 1)(x - 2)$.

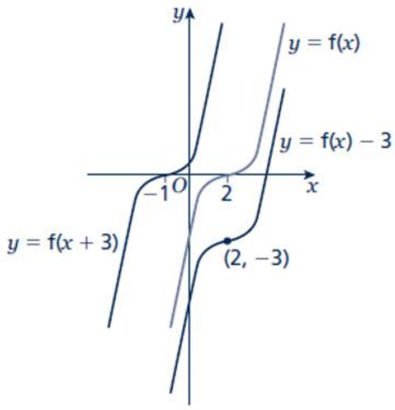
**b** On the same axes, sketch and label the graph of $y = \mathrm{f}\left(-\frac{1}{2}x\right)$.

## Answers

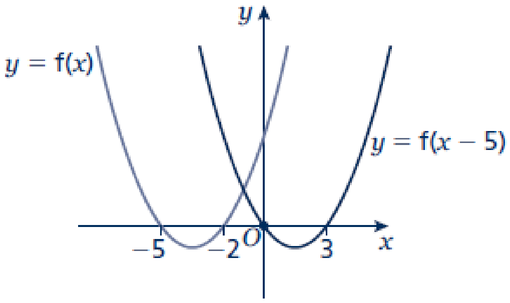
**1**




**2**



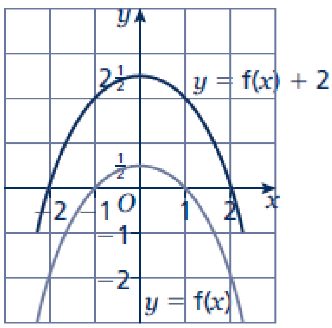

**3**



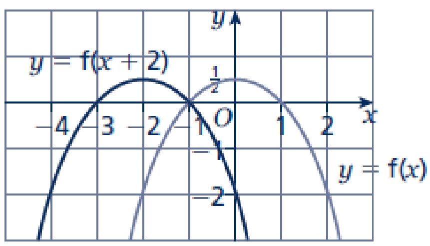

**4** $C_1$: $y = \mathrm{f}(x – 90°)$
$C_2$: $y = \mathrm{f}(x) – 2$

**5** $C_1$: $y = \mathrm{f}(x – 5)$
$C_2$: $y = \mathrm{f}(x) – 3$

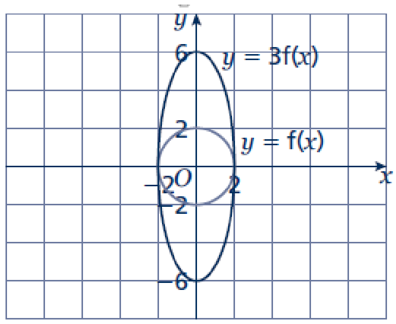
**6** **a**



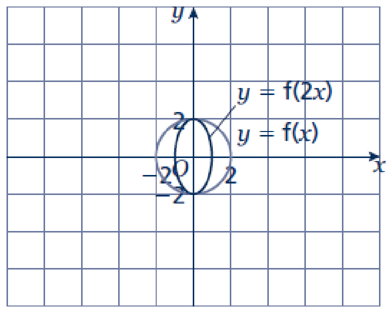

**b**



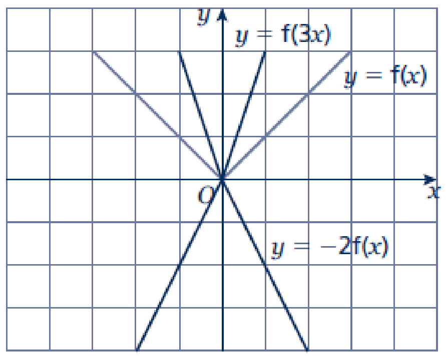
**7 a**



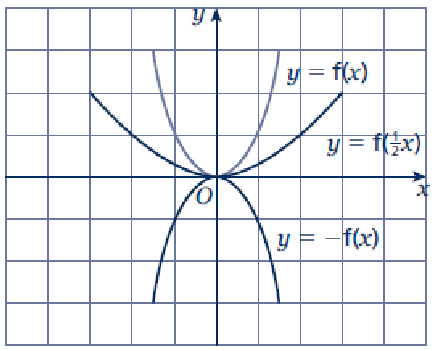

**b**



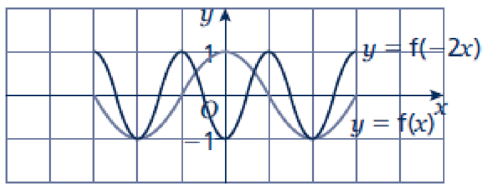

**8**




**9**



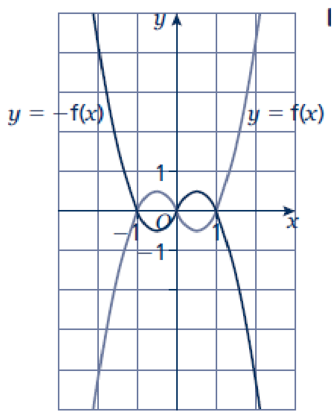

**10**



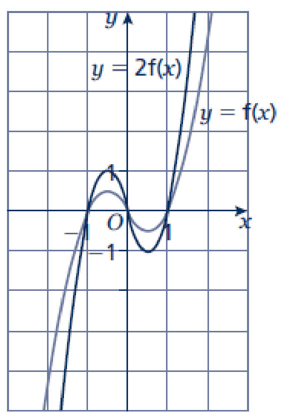

**11** $y = \mathrm{f}(2x)$

**12** $y = -2\mathrm{f}(2x)$ or $y = 2\mathrm{f}(-2x)$

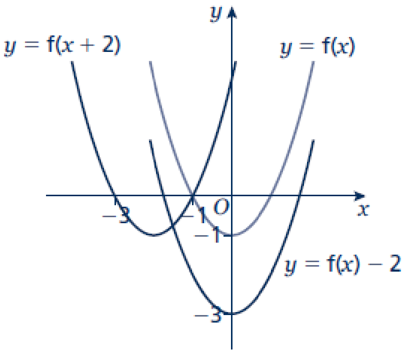
**13 a**



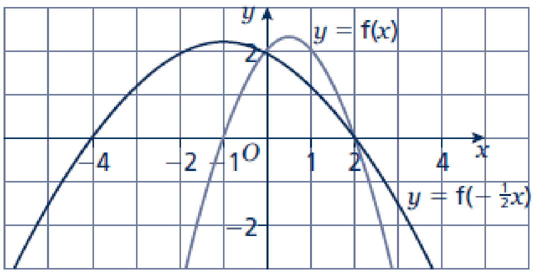

**b**



**14**




**15**

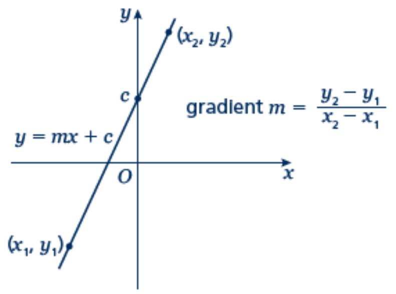


# Straight line graphs

**A LEVEL LINKS**
**Scheme of work:** 2a. Straight-line graphs, parallel/perpendicular, length and area problems

## Key points

- A straight line has the equation $y = mx + c$, where $m$ is the gradient and $c$ is the $y$-intercept (where $x = 0$).
- The equation of a straight line can be written in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers.
- When given the coordinates $(x_1, y_1)$ and $(x_2, y_2)$ of two points on a line the gradient is calculated using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$




## Examples

**Example 1** A straight line has gradient $-\frac{1}{2}$ and $y$-intercept 3.

Write the equation of the line in the form $ax + by + c = 0$.

| | |
|---|---|
| $m = -\frac{1}{2}$ and $c = 3$<br>So $y = -\frac{1}{2}x + 3$<br>$\frac{1}{2}x + y - 3 = 0$<br>$x + 2y - 6 = 0$ | **1** A straight line has equation $y = mx + c$. Substitute the gradient and $y$-intercept given in the question into this equation.<br>**2** Rearrange the equation so all the terms are on one side and 0 is on the other side.<br>**3** Multiply both sides by 2 to eliminate the denominator. |

**Example 2** Find the gradient and the $y$-intercept of the line with the equation $3y - 2x + 4 = 0$.

| | |
|---|---|
| $3y - 2x + 4 = 0$<br>$3y = 2x - 4$<br>$y = \frac{2}{3}x - \frac{4}{3}$<br>Gradient $= m = \frac{2}{3}$<br>$y$-intercept $= c = -\frac{4}{3}$ | **1** Make $y$ the subject of the equation.<br>**2** Divide all the terms by three to get the equation in the form $y = \ldots$<br>**3** In the form $y = mx + c$, the gradient is $m$ and the $y$-intercept is $c$. |

**Example 3** Find the equation of the line which passes through the point (5, 13) and has gradient 3.

| | |
|---|---|
| $m = 3$<br>$y = 3x + c$ | **1** Substitute the gradient given in the question into the equation of a straight line $y = mx + c$. |
| $13 = 3 \times 5 + c$ | **2** Substitute the coordinates $x = 5$ and $y = 13$ into the equation. |
| $13 = 15 + c$<br>$c = -2$ | **3** Simplify and solve the equation. |
| $y = 3x - 2$ | **4** Substitute $c = -2$ into the equation $y = 3x + c$ |

**Example 4** Find the equation of the line passing through the points with coordinates (2, 4) and (8, 7).

| | |
|---|---|
| $x_1 = 2$, $x_2 = 8$, $y_1 = 4$ and $y_2 = 7$<br>$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{8-2} = \frac{3}{6} = \frac{1}{2}$ | **1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line. |
| $y = \frac{1}{2}x + c$ | **2** Substitute the gradient into the equation of a straight line $y = mx + c$. |
| $4 = \frac{1}{2} \times 2 + c$ | **3** Substitute the coordinates of either point into the equation. |
| $c = 3$ | **4** Simplify and solve the equation. |
| $y = \frac{1}{2}x + 3$ | **5** Substitute $c = 3$ into the equation $y = \frac{1}{2}x + c$ |

## Practice

**1** Find the gradient and the $y$-intercept of the following equations.

**a** $y = 3x + 5$  **b** $y = -\frac{1}{2}x - 7$

**c** $2y = 4x - 3$  **d** $x + y = 5$

**e** $2x - 3y - 7 = 0$  **f** $5x + y - 4 = 0$

> **Hint**
> Rearrange the equations to the form $y = mx + c$

**2** Copy and complete the table, giving the equation of the line in the form $y = mx + c$.

| Gradient | $y$-intercept | Equation of the line |
|---|---|---|
| 5 | 0 | |
| –3 | 2 | |
| 4 | –7 | |

**3** Find, in the form $ax + by + c = 0$ where $a$, $b$ and $c$ are integers, an equation for each of the lines with the following gradients and $y$-intercepts.

**a** gradient $-\frac{1}{2}$, $y$-intercept –7

**b** gradient 2, $y$-intercept 0

**c** gradient $\frac{2}{3}$, $y$-intercept 4

**d** gradient –1.2, $y$-intercept –2

**4** Write an equation for the line which passes though the point (2, 5) and has gradient 4.

**5** Write an equation for the line which passes through the point (6, 3) and has gradient $-\frac{2}{3}$

**6** Write an equation for the line passing through each of the following pairs of points.

**a** (4, 5), (10, 17)

**b** (0, 6), (–4, 8)

**c** (–1, –7), (5, 23)

**d** (3, 10), (4, 7)

## Extend

**7** The equation of a line is $2y + 3x - 6 = 0$.
Write as much information as possible about this line.

# Answers

**1** **a** $m = 3$, $c = 5$

**b** $m = -\frac{1}{2}$, $c = -7$

**c** $m = 2$, $c = -\frac{3}{2}$

**d** $m = -1$, $c = 5$

**e** $m = \frac{2}{3}$, $c = -\frac{7}{3}$ or $-2\frac{1}{3}$

**f** $m = -5$, $c = 4$

**2**

| Gradient | $y$-intercept | Equation of the line |
|---|---|---|
| 5 | 0 | $y = 5x$ |
| –3 | 2 | $y = -3x + 2$ |
| 4 | –7 | $y = 4x - 7$ |

**3** **a** $x + 2y + 14 = 0$

**b** $2x - y = 0$

**c** $2x - 3y + 12 = 0$

**d** $6x + 5y + 10 = 0$

**4** $y = 4x - 3$

**5** $y = -\frac{2}{3}x + 7$

**6** **a** $y = 2x - 3$

**b** $y = -\frac{1}{2}x + 6$

**c** $y = 5x - 2$

**d** $y = -3x + 19$

**7** $y = -\frac{3}{2}x + 3$, the gradient is $-\frac{3}{2}$ and the $y$-intercept is 3.

The line intercepts the axes at (0, 3) and (2, 0).

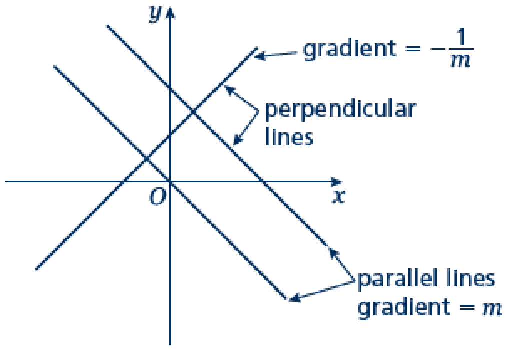
Students may sketch the line or give coordinates that lie on the line such as $\left(1, \frac{3}{2}\right)$ or $(4, -3)$.

# Parallel and perpendicular lines

**A LEVEL LINKS**
**Scheme of work:** 2a. Straight-line graphs, parallel/perpendicular, length and area problems

## Key points

- When lines are parallel they have the same gradient.
- A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.




## Examples

**Example 1** Find the equation of the line parallel to $y = 2x + 4$ which passes through the point (4, 9).

| | |
|---|---|
| $y = 2x + 4$<br>$m = 2$<br>$y = 2x + c$ | **1** As the lines are parallel they have the same gradient.<br>**2** Substitute $m = 2$ into the equation of a straight line $y = mx + c$. |
| $9 = 2 \times 4 + c$ | **3** Substitute the coordinates into the equation $y = 2x + c$ |
| $9 = 8 + c$<br>$c = 1$ | **4** Simplify and solve the equation. |
| $y = 2x + 1$ | **5** Substitute $c = 1$ into the equation $y = 2x + c$ |

**Example 2** Find the equation of the line perpendicular to $y = 2x - 3$ which passes through the point $(-2, 5)$.

| | |
|---|---|
| $y = 2x - 3$<br>$m = 2$<br>$-\frac{1}{m} = -\frac{1}{2}$ | **1** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$. |
| $y = -\frac{1}{2}x + c$ | **2** Substitute $m = -\frac{1}{2}$ into $y = mx + c$. |
| $5 = -\frac{1}{2} \times (-2) + c$ | **3** Substitute the coordinates $(-2, 5)$ into the equation $y = -\frac{1}{2}x + c$ |
| $5 = 1 + c$<br>$c = 4$ | **4** Simplify and solve the equation. |
| $y = -\frac{1}{2}x + 4$ | **5** Substitute $c = 4$ into $y = -\frac{1}{2}x + c$. |

**Example 3** A line passes through the points (0, 5) and (9, −1).
Find the equation of the line which is perpendicular to the line and passes through its midpoint.

| | |
|---|---|
| $x_1 = 0$, $x_2 = 9$, $y_1 = 5$ and $y_2 = -1$<br>$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{9-0}$<br>$= \frac{-6}{9} = -\frac{2}{3}$ | **1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line. |
| $-\frac{1}{m} = \frac{3}{2}$ | **2** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$. |
| $y = \frac{3}{2}x + c$ | **3** Substitute the gradient into the equation $y = mx + c$. |
| Midpoint = $\left(\frac{0+9}{2}, \frac{5+(-1)}{2}\right) = \left(\frac{9}{2}, 2\right)$ | **4** Work out the coordinates of the midpoint of the line. |
| $2 = \frac{3}{2} \times \frac{9}{2} + c$ | **5** Substitute the coordinates of the midpoint into the equation. |
| $c = -\frac{19}{4}$ | **6** Simplify and solve the equation. |
| $y = \frac{3}{2}x - \frac{19}{4}$ | **7** Substitute $c = -\frac{19}{4}$ into the equation $y = \frac{3}{2}x + c$. |

## Practice

**1** Find the equation of the line parallel to each of the given lines and which passes through each of the given points.

**a** $y = 3x + 1$ (3, 2)
**b** $y = 3 - 2x$ (1, 3)
**c** $2x + 4y + 3 = 0$ (6, –3)
**d** $2y - 3x + 2 = 0$ (8, 20)

**2** Find the equation of the line perpendicular to $y = \frac{1}{2}x - 3$ which passes through the point (–5, 3).

**Hint**
If $m = \frac{a}{b}$ then the negative reciprocal $-\frac{1}{m} = -\frac{b}{a}$

**3** Find the equation of the line perpendicular to each of the given lines and which passes through each of the given points.

**a** $y = 2x - 6$ (4, 0)
**b** $y = -\frac{1}{3}x + \frac{1}{2}$ (2, 13)
**c** $x - 4y - 4 = 0$ (5, 15)
**d** $5y + 2x - 5 = 0$ (6, 7)

**4** In each case find an equation for the line passing through the origin which is also perpendicular to the line joining the two points given.

**a** (4, 3), (–2, –9) **b** (0, 3), (–10, 8)

## Extend

**5** Work out whether these pairs of lines are parallel, perpendicular or neither.

**a** $y = 2x + 3$
$y = 2x - 7$

**b** $y = 3x$
$2x + y - 3 = 0$

**c** $y = 4x - 3$
$4y + x = 2$

**d** $3x - y + 5 = 0$
$x + 3y = 1$

**e** $2x + 5y - 1 = 0$
$y = 2x + 7$

**f** $2x - y = 6$
$6x - 3y + 3 = 0$

**6** The straight line $\mathbf{L_1}$ passes through the points $A$ and $B$ with coordinates (–4, 4) and (2, 1), respectively.

**a** Find the equation of $\mathbf{L_1}$ in the form $ax + by + c = 0$

The line $\mathbf{L_2}$ is parallel to the line $\mathbf{L_1}$ and passes through the point $C$ with coordinates (–8, 3).

**b** Find the equation of $\mathbf{L_2}$ in the form $ax + by + c = 0$

The line $\mathbf{L_3}$ is perpendicular to the line $\mathbf{L_1}$ and passes through the origin.

**c** Find an equation of $\mathbf{L_3}$

## Answers

**1** **a** $y = 3x - 7$ **b** $y = -2x + 5$

**c** $y = -\frac{1}{2}x$ **d** $y = \frac{3}{2}x + 8$

**2** $y = -2x - 7$

**3** **a** $y = -\frac{1}{2}x + 2$ **b** $y = 3x + 7$

**c** $y = -4x + 35$ **d** $y = \frac{5}{2}x - 8$

**4** **a** $y = -\frac{1}{2}x$ **b** $y = 2x$

**5** **a** Parallel **b** Neither **c** Perpendicular

**d** Perpendicular **e** Neither **f** Parallel

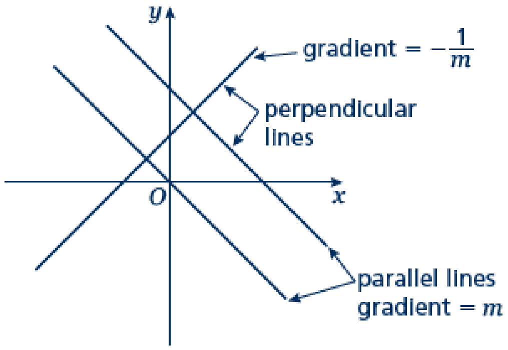
**6** **a** $x + 2y - 4 = 0$ **b** $x + 2y + 2 = 0$ **c** $y = 2x$

# Parallel and perpendicular lines

**A LEVEL LINKS**
**Scheme of work:** 2a. Straight-line graphs, parallel/perpendicular, length and area problems

## Key points

- When lines are parallel they have the same gradient.
- A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.




## Examples

**Example 1** Find the equation of the line parallel to $y = 2x + 4$ which passes through the point (4, 9).

| | |
|---|---|
| $y = 2x + 4$<br>$m = 2$<br>$y = 2x + c$<br><br>$9 = 2 \times 4 + c$<br><br>$9 = 8 + c$<br>$c = 1$<br>$y = 2x + 1$ | **1** As the lines are parallel they have the same gradient.<br>**2** Substitute $m = 2$ into the equation of a straight line $y = mx + c$.<br>**3** Substitute the coordinates into the equation $y = 2x + c$<br>**4** Simplify and solve the equation.<br><br>**5** Substitute $c = 1$ into the equation $y = 2x + c$ |

**Example 2** Find the equation of the line perpendicular to $y = 2x - 3$ which passes through the point $(-2, 5)$.

| | |
|---|---|
| $y = 2x - 3$<br>$m = 2$<br>$-\frac{1}{m} = -\frac{1}{2}$<br>$y = -\frac{1}{2}x + c$<br>$5 = -\frac{1}{2} \times (-2) + c$<br><br>$5 = 1 + c$<br>$c = 4$<br>$y = -\frac{1}{2}x + 4$ | **1** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.<br>**2** Substitute $m = -\frac{1}{2}$ into $y = mx + c$.<br>**3** Substitute the coordinates $(-2, 5)$ into the equation $y = -\frac{1}{2}x + c$<br>**4** Simplify and solve the equation.<br><br>**5** Substitute $c = 4$ into $y = -\frac{1}{2}x + c$. |

**Example 3** A line passes through the points (0, 5) and (9, −1).
Find the equation of the line which is perpendicular to the line and passes through its midpoint.

| | |
|---|---|
| $x_1 = 0$, $x_2 = 9$, $y_1 = 5$ and $y_2 = -1$<br>$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{9-0}$<br>$= \frac{-6}{9} = -\frac{2}{3}$ | **1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line. |
| $-\frac{1}{m} = \frac{3}{2}$ | **2** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$. |
| $y = \frac{3}{2}x + c$ | **3** Substitute the gradient into the equation $y = mx + c$. |
| Midpoint = $\left(\frac{0+9}{2}, \frac{5+(-1)}{2}\right) = \left(\frac{9}{2}, 2\right)$ | **4** Work out the coordinates of the midpoint of the line. |
| $2 = \frac{3}{2} \times \frac{9}{2} + c$ | **5** Substitute the coordinates of the midpoint into the equation. |
| $c = -\frac{19}{4}$ | **6** Simplify and solve the equation. |
| $y = \frac{3}{2}x - \frac{19}{4}$ | **7** Substitute $c = -\frac{19}{4}$ into the equation $y = \frac{3}{2}x + c$. |

## Practice

**1** Find the equation of the line parallel to each of the given lines and which passes through each of the given points.

**a** $y = 3x + 1$ (3, 2)
**b** $y = 3 - 2x$ (1, 3)
**c** $2x + 4y + 3 = 0$ (6, –3)
**d** $2y - 3x + 2 = 0$ (8, 20)

**2** Find the equation of the line perpendicular to $y = \frac{1}{2}x - 3$ which passes through the point (–5, 3).

**Hint**
If $m = \frac{a}{b}$ then the negative reciprocal $-\frac{1}{m} = -\frac{b}{a}$

**3** Find the equation of the line perpendicular to each of the given lines and which passes through each of the given points.

**a** $y = 2x - 6$ (4, 0)
**b** $y = -\frac{1}{3}x + \frac{1}{2}$ (2, 13)
**c** $x - 4y - 4 = 0$ (5, 15)
**d** $5y + 2x - 5 = 0$ (6, 7)

**4** In each case find an equation for the line passing through the origin which is also perpendicular to the line joining the two points given.

**a** (4, 3), (–2, –9) **b** (0, 3), (–10, 8)

## Extend

**5** Work out whether these pairs of lines are parallel, perpendicular or neither.

**a** $y = 2x + 3$
$y = 2x - 7$

**b** $y = 3x$
$2x + y - 3 = 0$

**c** $y = 4x - 3$
$4y + x = 2$

**d** $3x - y + 5 = 0$
$x + 3y = 1$

**e** $2x + 5y - 1 = 0$
$y = 2x + 7$

**f** $2x - y = 6$
$6x - 3y + 3 = 0$

**6** The straight line $\mathbf{L}_1$ passes through the points $A$ and $B$ with coordinates (–4, 4) and (2, 1), respectively.

**a** Find the equation of $\mathbf{L}_1$ in the form $ax + by + c = 0$

The line $\mathbf{L}_2$ is parallel to the line $\mathbf{L}_1$ and passes through the point $C$ with coordinates (–8, 3).

**b** Find the equation of $\mathbf{L}_2$ in the form $ax + by + c = 0$

The line $\mathbf{L}_3$ is perpendicular to the line $\mathbf{L}_1$ and passes through the origin.

**c** Find an equation of $\mathbf{L}_3$

## Answers

**1** **a** $y = 3x - 7$ **b** $y = -2x + 5$

**c** $y = -\frac{1}{2}x$ **d** $y = \frac{3}{2}x + 8$

**2** $y = -2x - 7$

**3** **a** $y = -\frac{1}{2}x + 2$ **b** $y = 3x + 7$

**c** $y = -4x + 35$ **d** $y = \frac{5}{2}x - 8$

**4** **a** $y = -\frac{1}{2}x$ **b** $y = 2x$

**5** **a** Parallel **b** Neither **c** Perpendicular

**d** Perpendicular **e** Neither **f** Parallel

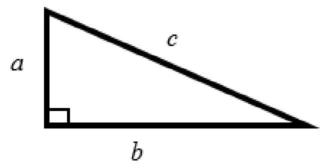
**6** **a** $x + 2y - 4 = 0$ **b** $x + 2y + 2 = 0$ **c** $y = 2x$

# Pythagoras’ theorem

**A LEVEL LINKS**
**Scheme of work:** 2a. Straight-line graphs, parallel/perpendicular, length and area problems

## Key points

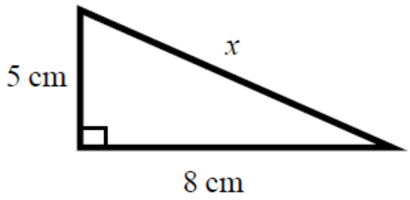
- In a right-angled triangle the longest side is called the hypotenuse.
- Pythagoras’ theorem states that for a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.
  = $a^2 + b^2$

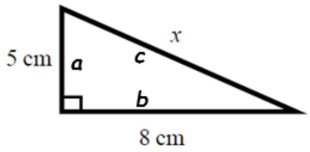



$c^2$

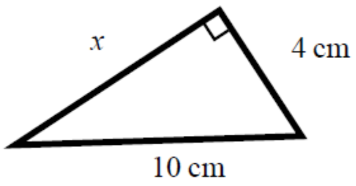
## Examples

**Example 1** Calculate the length of the hypotenuse.
Give your answer to 3 significant figures.




| | |
|---|---|
| $c^2 = a^2 + b^2$<br><br> | **1** Always start by stating the formula for Pythagoras’ theorem and labelling the hypotenuse $c$ and the other two sides $a$ and $b$. |
| $x^2 = 5^2 + 8^2$<br>$x^2 = 25 + 64$<br>$x^2 = 89$<br>$x = \sqrt{89}$<br><br>$x = 9.433\ 981\ 13...$<br>$x = 9.43$ cm | **2** Substitute the values of $a$, $b$ and $c$ into the formula for Pythagoras' theorem.<br>**3** Use a calculator to find the square root.<br>**4** Round your answer to 3 significant figures and write the units with your answer. |

**Example 2** Calculate the length $x$.
Give your answer in surd form.

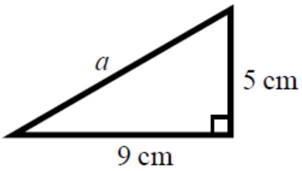



| | |
|---|---|
| $c^2 = a^2 + b^2$<br><br>$10^2 = x^2 + 4^2$<br>$100 = x^2 + 16$<br>$x^2 = 84$<br>$x = \sqrt{84}$<br>$x = 2\sqrt{21}$ cm | **1** Always start by stating the formula for Pythagoras' theorem.<br>**2** Substitute the values of $a$, $b$ and $c$ into the formula for Pythagoras' theorem.<br><br>**3** Simplify the surd where possible and write the units in your answer. |

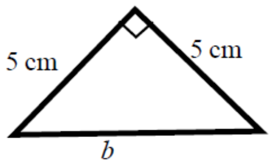
## Practice

**1** Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

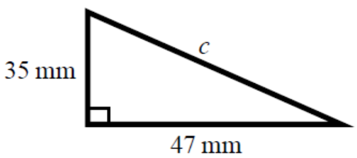
**a**



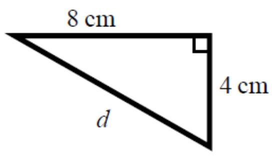

**b**




**c**

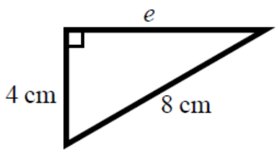



**d**

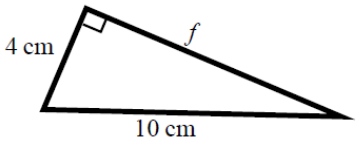



**2** Work out the length of the unknown side in each triangle.
Give your answers in surd form.

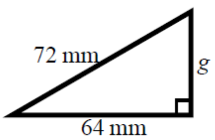
**a**



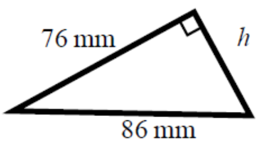

**b**




**c**

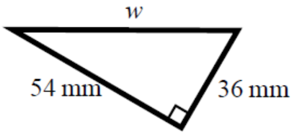



**d**

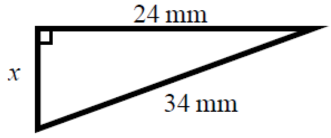


**3** Work out the length of the unknown side in each triangle.
Give your answers in surd form.

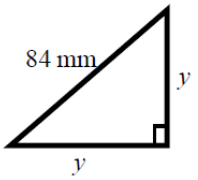
**a**



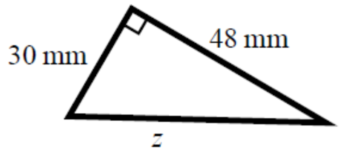

**b**




**c**




**d**




**4** A rectangle has length 84 mm and width 45 mm.
Calculate the length of the diagonal of the rectangle.
Give your answer correct to 3 significant figures.

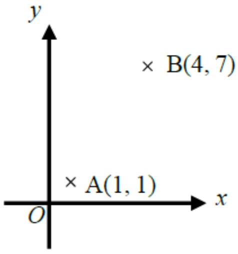
> **Hint**
>
> Draw a sketch of the rectangle.

## Extend

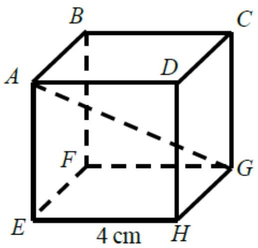
**5** A yacht is 40 km due North of a lighthouse.
A rescue boat is 50 km due East of the same lighthouse.
Work out the distance between the yacht and the rescue boat.
Give your answer correct to 3 significant figures.

> **Hint**
>
> Draw a diagram using the information given in the question.

**6** Points A and B are shown on the diagram.
Work out the length of the line AB.
Give your answer in surd form.




**7** A cube has length 4 cm.
Work out the length of the diagonal *AG*.
Give your answer in surd form.



## Answers

**1** **a** 10.3 cm **b** 7.07 cm

**c** 58.6 mm **d** 8.94 cm

**2** **a** $4\sqrt{3}$ cm **b** $2\sqrt{21}$ cm

**c** $8\sqrt{17}$ mm **d** $18\sqrt{5}$ mm

**3** **a** $18\sqrt{13}$ mm **b** $2\sqrt{145}$ mm

**c** $42\sqrt{2}$ mm **d** $6\sqrt{89}$ mm

**4** 95.3 mm

**5** 64.0 km

**6** $3\sqrt{5}$ units

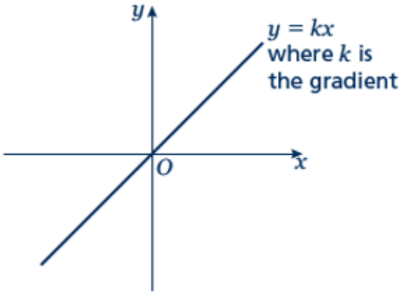
**7** $4\sqrt{3}$ cm

# Proportion

**A LEVEL LINKS**
**Scheme of work:** 2a. Straight-line graphs, parallel/perpendicular, length and area problems

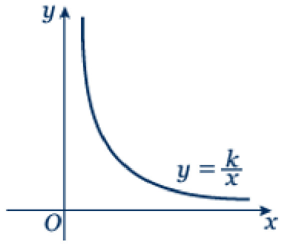
## Key points

- Two quantities are in direct proportion when, as one quantity increases, the other increases at the same rate. Their ratio remains the same.
- '$y$ is directly proportional to $x$' is written as $y \propto x$. If $y \propto x$ then $y = kx$, where $k$ is a constant.
- When $x$ is directly proportional to $y$, the graph is a straight line passing through the origin.




- Two quantities are in inverse proportion when, as one quantity increases, the other decreases at the same rate.
- '$y$ is inversely proportional to $x$' is written as $y \propto \frac{1}{x}$.

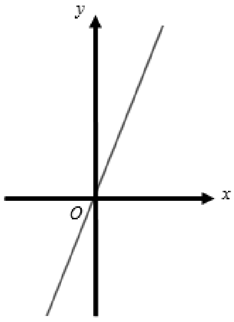
  If $y \propto \frac{1}{x}$ then $y = \frac{k}{x}$, where $k$ is a constant.
- When $x$ is inversely proportional to $y$ the graph is the same shape as the graph of $y = \frac{1}{x}$




## Examples

**Example 1** $y$ is directly proportional to $x$.
When $y = 16$, $x = 5$.
**a** Find $x$ when $y = 30$.
**b** Sketch the graph of the formula.

| | |
|---|---|
| **a** $y \propto x$ | **1** Write $y$ is directly proportional to $x$, using the symbol $\propto$. |
| $y = kx$ | **2** Write the equation using $k$. |
| $16 = k \times 5$ | **3** Substitute $y = 16$ and $x = 5$ into $y = kx$. |
| $k = 3.2$ | **4** Solve the equation to find $k$. |
| $y = 3.2x$ | **5** Substitute the value of $k$ back into the equation $y = kx$. |
| When $y = 30$,<br>$30 = 3.2 \times x$<br>$x = 9.375$ | **6** Substitute $y = 30$ into $y = 3.2x$ and solve to find $x$ when $y = 30$. |

| | |
|---|---|
| **b**  | **7** The graph of $y = 3.2x$ is a straight line passing through (0, 0) with a gradient of 3.2. |

**Example 2** $y$ is directly proportional to $x^2$.
When $x = 3$, $y = 45$.
**a** Find $y$ when $x = 5$.
**b** Find $x$ when $y = 20$.

| | |
|---|---|
| **a** $y \propto x^2$ | **1** Write $y$ is directly proportional to $x^2$, using the symbol $\propto$. |
| $y = kx^2$<br>$45 = k \times 3^2$ | **2** Write the equation using $k$.<br>**3** Substitute $y = 45$ and $x = 3$ into $y = kx^2$. |
| $k = 5$<br>$y = 5x^2$ | **4** Solve the equation to find $k$.<br>**5** Substitute the value of $k$ back into the equation $y = kx^2$. |
| When $x = 5$,<br>$y = 5 \times 5^2$<br>$y = 125$ | **6** Substitute $x = 5$ into $y = 5x^2$ and solve to find $y$ when $x = 5$. |
| **b** $20 = 5 \times x^2$<br>$x^2 = 4$<br>$x = \pm 2$ | **7** Substitute $y = 20$ into $y = 5x^2$ and solve to find $x$ when $y = 4$. |

**Example 3** $P$ is inversely proportional to $Q$.
When $P = 100$, $Q = 10$.
Find $Q$ when $P = 20$.

| | |
|---|---|
| $P \propto \frac{1}{Q}$ | **1** Write $P$ is inversely proportional to $Q$, using the symbol $\propto$. |
| $P = \frac{k}{Q}$ | **2** Write the equation using $k$. |
| $100 = \frac{k}{10}$ | **3** Substitute $P = 100$ and $Q = 10$. |
| $k = 1000$ | **4** Solve the equation to find $k$. |
| $P = \frac{1000}{Q}$ | **5** Substitute the value of $k$ into $P = \frac{k}{Q}$ |
| $20 = \frac{1000}{Q}$<br>$Q = \frac{1000}{20} = 50$ | **6** Substitute $P = 20$ into $P = \frac{1000}{Q}$ and solve to find $Q$ when $P = 20$. |

## Practice

> **Hint**
>
> Substitute the values given for $P$ and $h$ into the formula to calculate $k$.

**1** Paul gets paid an hourly rate. The amount of pay (£$P$) is directly proportional to the number of hours ($h$) he works.
When he works 8 hours he is paid £56.
If Paul works for 11 hours, how much is he paid?

**2** $x$ is directly proportional to $y$.
$x = 35$ when $y = 5$.

**a** Find a formula for $x$ in terms of $y$.

**b** Sketch the graph of the formula.

**c** Find $x$ when $y = 13$.

**d** Find $y$ when $x = 63$.

**3** $Q$ is directly proportional to the square of $Z$.
$Q = 48$ when $Z = 4$.

**a** Find a formula for $Q$ in terms of $Z$.

**b** Sketch the graph of the formula.

**c** Find $Q$ when $Z = 5$.

**d** Find $Z$ when $Q = 300$.

**4** $y$ is directly proportional to the square of $x$.
$x = 2$ when $y = 10$.

**a** Find a formula for $y$ in terms of $x$.

**b** Sketch the graph of the formula.

**c** Find $x$ when $y = 90$.

**5** $B$ is directly proportional to the square root of $C$.
$C = 25$ when $B = 10$.

**a** Find $B$ when $C = 64$.

**b** Find $C$ when $B = 20$.

**6** $C$ is directly proportional to $D$.
$C = 100$ when $D = 150$.
Find $C$ when $D = 450$.

**7** $y$ is directly proportional to $x$.
$x = 27$ when $y = 9$.
Find $x$ when $y = 3.7$.

**8** $m$ is proportional to the cube of $n$.
$m = 54$ when $n = 3$.
Find $n$ when $m = 250$.

## Extend

**9** $s$ is inversely proportional to $t$.

**a** Given that $s = 2$ when $t = 2$, find a formula for $s$ in terms of $t$.

**b** Sketch the graph of the formula.

**c** Find $t$ when $s = 1$.

**10** $a$ is inversely proportional to $b$.
$a = 5$ when $b = 20$.

**a** Find $a$ when $b = 50$.

**b** Find $b$ when $a = 10$.

**11** $v$ is inversely proportional to $w$.
$w = 4$ when $v = 20$.

**a** Find a formula for $v$ in terms of $w$.

**b** Sketch the graph of the formula.

**c** Find $w$ when $v = 2$.

**12** $L$ is inversely proportional to $W$.
$L = 12$ when $W = 3$.
Find $W$ when $L = 6$.

**13** $s$ is inversely proportional to $t$.
$s = 6$ when $t = 12$.

**a** Find $s$ when $t = 3$.

**b** Find $t$ when $s = 18$.

**14** $y$ is inversely proportional to $x^2$.
$y = 4$ when $x = 2$.
Find $y$ when $x = 4$.

**15** $y$ is inversely proportional to the square root of $x$.
$x = 25$ when $y = 1$.
Find $x$ when $y = 5$.

**16** $a$ is inversely proportional to $b$.
$a = 0.05$ when $b = 4$.

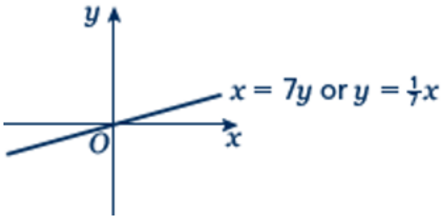
**a** Find $a$ when $b = 2$.

**b** Find $b$ when $a = 2$.

## Answers

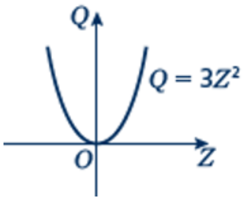
**1** £77

**2** **a** $x = 7y$ **b**



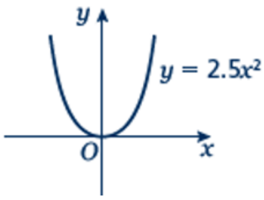

**c** 91 **d** 9

**3** **a** $Q = 3Z^2$ **b**




**c** 75 **d** ±10

**4** **a** $y = 2.5x^2$ **b**




**c** ±6

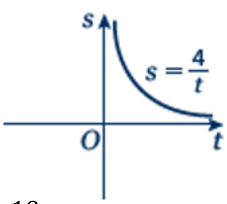
**5** **a** 16 **b** 100

**6** 300

**7** 11.1

**8** 5

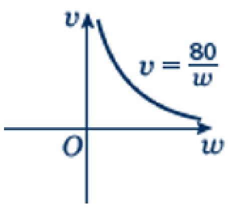
**9** **a** $s = \frac{4}{t}$ **b**




**c** 4

**10** **a** 2 **b** 10

**11** **a** $v = \frac{80}{w}$ **b**




**c** 40

**12** 6

**13** **a** 24 **b** 4

**14** 1

**15** 1

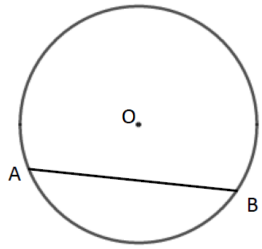
**16** **a** 0.1 **b** 0.1

# Circle theorems

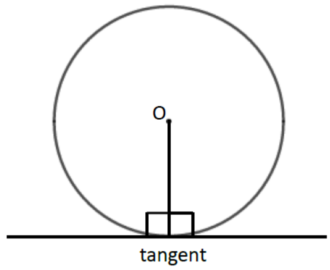
**A LEVEL LINKS**
**Scheme of work:** 2b. Circles – equation of a circle, geometric problems on a grid

## Key points

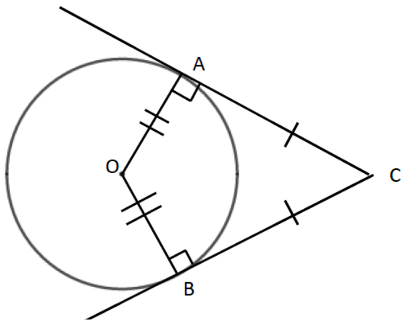
- A chord is a straight line joining two points on the circumference of a circle.
  So AB is a chord.



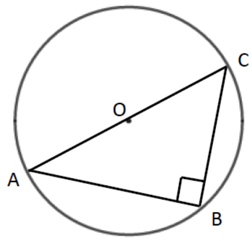

- A tangent is a straight line that touches the circumference of a circle at only one point.
  The angle between a tangent and the radius is 90°.



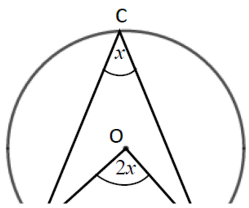

- Two tangents on a circle that meet at a point outside the circle are equal in length.
  So AC = BC.



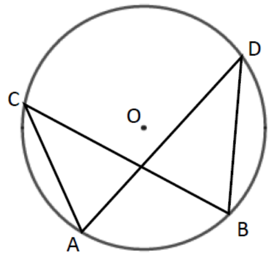

- The angle in a semicircle is a right angle.
  So angle ABC = 90°.




- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
  So angle AOB = 2 × angle ACB.

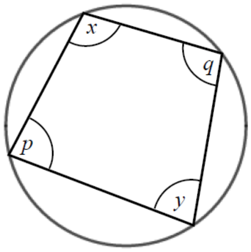



- Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
  So angle ACB = angle ADB and

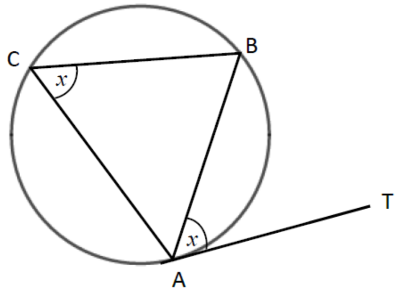


angle CAD = angle CBD.

- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle. Opposite angles in a cyclic quadrilateral total 180°. So $x + y = 180°$ and $p + q = 180°$.

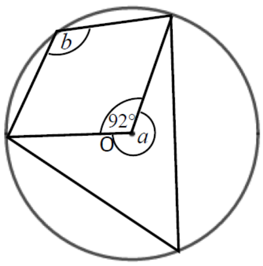



- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem. So angle BAT = angle ACB.



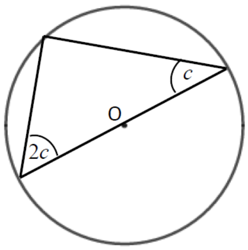

## Examples

**Example 1** Work out the size of each angle marked with a letter. Give reasons for your answers.



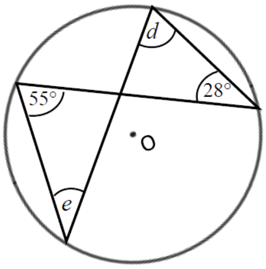

| | |
|---|---|
| Angle $a = 360° - 92°$<br>$= 268°$<br>as the angles in a full turn total 360°.<br><br>Angle $b = 268° \div 2$<br>$= 134°$<br>as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference. | **1** The angles in a full turn total 360°.<br><br><br>**2** Angles $a$ and $b$ are subtended by the same arc, so angle $b$ is half of angle $a$. |

**Example 2** Work out the size of the angles in the triangle. Give reasons for your answers.



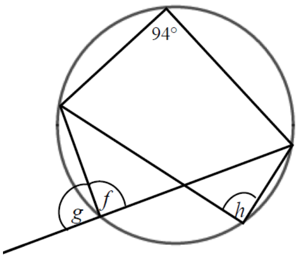
| | |
|---|---|
| Angles are 90°, $2c$ and $c$.<br><br>$90° + 2c + c = 180°$<br>$90° + 3c = 180°$<br>$3c = 90°$<br>$c = 30°$<br>$2c = 60°$<br><br>The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°. | **1** The angle in a semicircle is a right angle.<br>**2** Angles in a triangle total 180°.<br>**3** Simplify and solve the equation. |

**Example 3** Work out the size of each angle marked with a letter.
Give reasons for your answers.




| | |
|---|---|
| Angle $d$ = 55° as angles subtended by the same arc are equal.<br><br>Angle $e$ = 28° as angles subtended by the same arc are equal. | **1** Angles subtended by the same arc are equal so angle 55° and angle $d$ are equal.<br>**2** Angles subtended by the same arc are equal so angle 28° and angle $e$ are equal. |

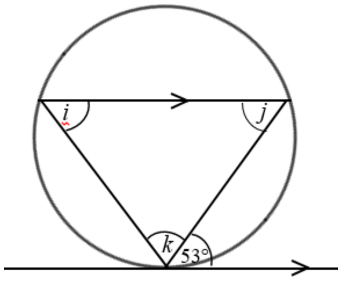
**Example 4** Work out the size of each angle marked with a letter.
Give reasons for your answers.




| | |
|---|---|
| Angle $f = 180° - 94°$<br>$= 86°$<br>as opposite angles in a cyclic quadrilateral total 180°. | **1** Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle $f$ total 180°.<br><br>*(continued on next page)* |

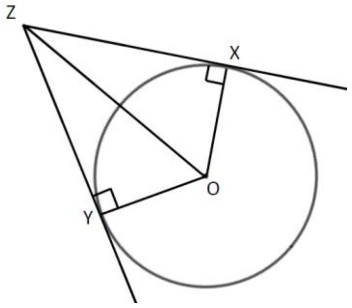
| | |
|---|---|
| Angle $g = 180° - 86°$<br>$= 84°$<br>as angles on a straight line total 180°.<br><br>Angle $h$ = angle $f$ = 86° as angles subtended by the same arc are equal. | **2** Angles on a straight line total 180° so angle $f$ and angle $g$ total 180°.<br><br>**3** Angles subtended by the same arc are equal so angle $f$ and angle $h$ are equal. |

**Example 5** Work out the size of each angle marked with a letter. Give reasons for your answers.




| | |
|---|---|
| Angle $i$ = 53° because of the alternate segment theorem.<br><br>Angle $j$ = 53° because it is the alternate angle to 53°.<br><br>Angle $k = 180° - 53° - 53°$<br>$= 74°$<br>as angles in a triangle total 180°. | **1** The angle between a tangent and chord is equal to the angle in the alternate segment.<br>**2** As there are two parallel lines, angle 53° is equal to angle $j$ because they are alternate angles.<br>**3** The angles in a triangle total 180°, so $i + j + k = 180°$. |

**Example 6** XZ and YZ are two tangents to a circle with centre O. Prove that triangles XZO and YZO are congruent.

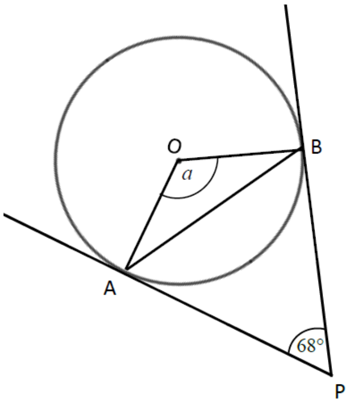



| | |
|---|---|
| Angle OXZ = 90° and angle OYZ = 90° as the angles in a semicircle are right angles.<br><br>OZ is a common line and is the hypotenuse in both triangles.<br><br>OX = OY as they are radii of the same circle.<br><br>So triangles XZO and YZO are congruent, RHS. | For two triangles to be congruent you need to show one of the following.<br>• All three corresponding sides are equal (SSS).<br>• Two corresponding sides and the included angle are equal (SAS).<br>• One side and two corresponding angles are equal (ASA).<br>• A right angle, hypotenuse and a shorter side are equal (RHS). |

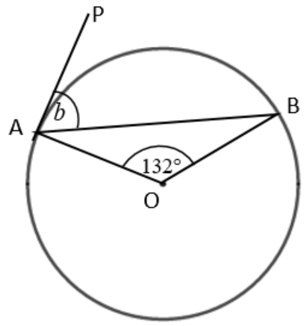
# Practice

**1** Work out the size of each angle marked with a letter.
Give reasons for your answers.

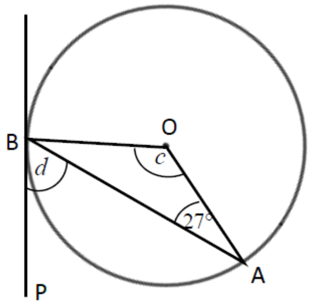
**a**



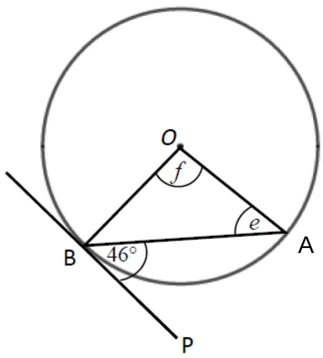

**b**



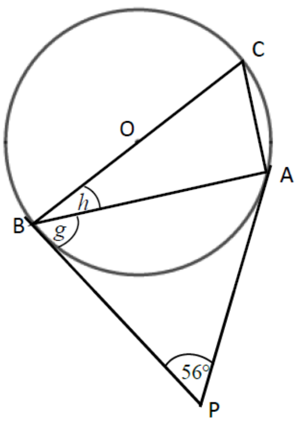

**c**




**d**

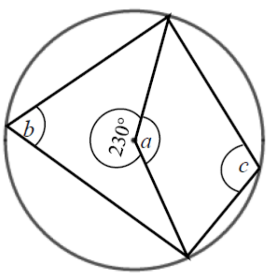



**e**

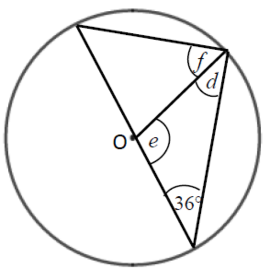



**2** Work out the size of each angle marked with a letter.
Give reasons for your answers.

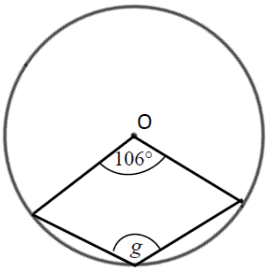
**a**




**b**



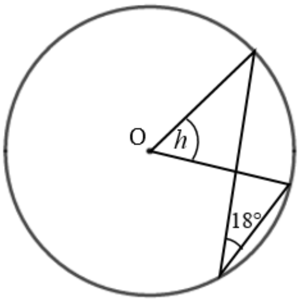
**c**




**Hint**

The reflex angle at point O and angle *g* are subtended by the same arc. So the reflex angle is twice the size of angle *g*.

**d**

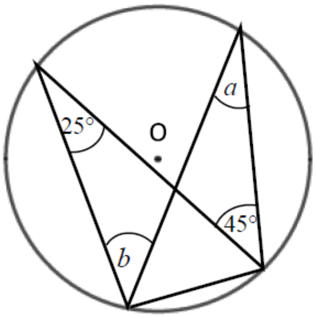



**Hint**

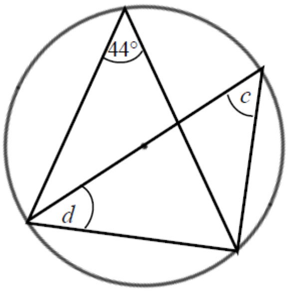
Angle 18° and angle *h* are subtended by the same arc.

**3** Work out the size of each angle marked with a letter.
Give reasons for your answers.

**a**



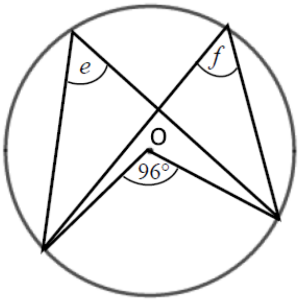

**b**



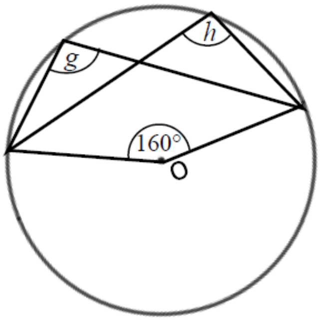

**Hint**

One of the angles is in a semicircle.

**c**

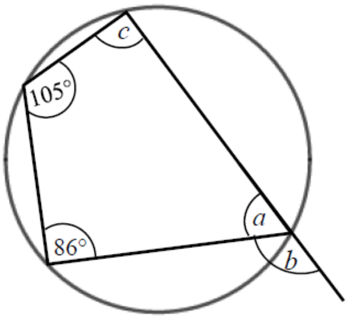



**d**



**4** Work out the size of each angle marked with a letter.
Give reasons for your answers.

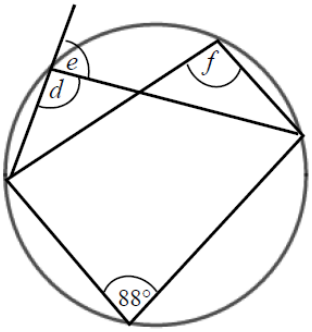
**a**



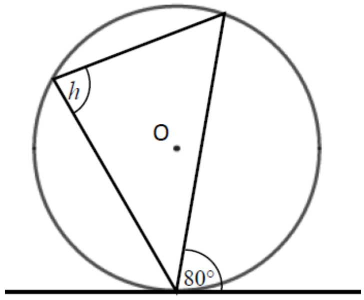

**Hint**

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

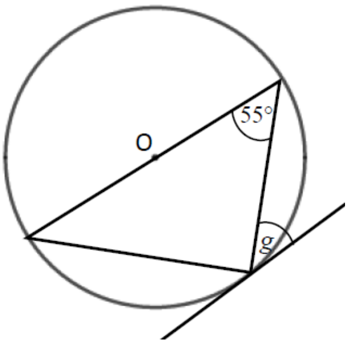
**b**




**c**




**d**




**Hint**

One of the angles is in a semicircle.

## Extend

**5** Prove the alternate segment theorem.

# Answers

**1** **a** $a = 112°$, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360°.

**b** $b = 66°$, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.

**c** $c = 126°$, triangle OAB is isosceles.
$d = 63°$, Angle OBP = 90° as BP is tangent to the circle.

**d** $e = 44°$, the triangle is isosceles, so angles $e$ and angle OBA are equal. The angle OBP = 90° as BP is tangent to the circle.
$f = 92°$, the triangle is isosceles.

**e** $g = 62°$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
$h = 28°$, the angle OBP = 90°.

**2** **a** $a = 130°$, angles in a full turn total 360°.
$b = 65°$, the angle at the centre of a circle is twice the angle at the circumference.
$c = 115°$, opposite angles in a cyclic quadrilateral total 180°.

**b** $d = 36°$, isosceles triangle.
$e = 108°$, angles in a triangle total 180°.
$f = 54°$, angle in a semicircle is 90°.

**c** $g = 127°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.

**d** $h = 36°$, the angle at the centre of a circle is twice the angle at the circumference.

**3** **a** $a = 25°$, angles in the same segment are equal.
$b = 45°$, angles in the same segment are equal.

**b** $c = 44°$, angles in the same segment are equal.
$d = 46°$, the angle in a semicircle is 90° and the angles in a triangle total 180°.

**c** $e = 48°$, the angle at the centre of a circle is twice the angle at the circumference.
$f = 48°$, angles in the same segment are equal.

**d** $g = 100°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.
$h = 100°$, angles in the same segment are equal.

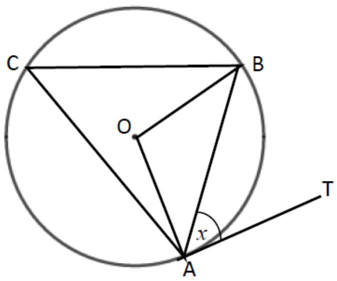
**4** **a** $a = 75°$, opposite angles in a cyclic quadrilateral total 180°.
$b = 105°$, angles on a straight line total 180°.
$c = 94°$, opposite angles in a cyclic quadrilateral total 180°.

**b** $d = 92°$, opposite angles in a cyclic quadrilateral total 180°.
$e = 88°$, angles on a straight line total 180°.
$f = 92°$, angles in the same segment are equal.

**c** $h = 80°$, alternate segment theorem.

**d** $g = 35°$, alternate segment theorem and the angle in a semicircle is 90°.

**5** Angle BAT = $x$.




Angle OAB = 90° − $x$ because the angle between the tangent and the radius is 90°.

OA = OB because radii are equal.

Angle OAB = angle OBA because the base of isosceles triangles are equal.

Angle AOB = 180° − (90° − $x$) − (90° − $x$) = 2$x$ because angles in a triangle total 180°.

Angle ACB = 2$x$ ÷ 2 = $x$ because the angle at the centre is twice the angle at the circumference.

# Rearranging equations

**A LEVEL LINKS**
**Scheme of work:** 6a. Definition, differentiating polynomials, second derivatives
**Textbook:** Pure Year 1, 12.1 Gradients of curves

## Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

## Examples

**Example 1** Make $t$ the subject of the formula $v = u + at$.

| | |
|---|---|
| $v = u + at$<br><br>$v - u = at$<br>$t = \dfrac{v-u}{a}$ | **1** Get the terms containing $t$ on one side and everything else on the other side.<br>**2** Divide throughout by $a$. |

**Example 2** Make $t$ the subject of the formula $r = 2t - \pi t$.

| | |
|---|---|
| $r = 2t - \pi t$<br><br>$r = t(2 - \pi)$<br>$t = \dfrac{r}{2-\pi}$ | **1** All the terms containing $t$ are already on one side and everything else is on the other side.<br>**2** Factorise as $t$ is a common factor.<br>**3** Divide throughout by $2 - \pi$. |

**Example 3** Make $t$ the subject of the formula $\dfrac{t+r}{5} = \dfrac{3t}{2}$.

| | |
|---|---|
| $\dfrac{t+r}{5} = \dfrac{3t}{2}$<br>$2t + 2r = 15t$<br>$2r = 13t$<br>$t = \dfrac{2r}{13}$ | **1** Remove the fractions first by multiplying throughout by 10.<br>**2** Get the terms containing $t$ on one side and everything else on the other side and simplify.<br>**3** Divide throughout by 13. |

edexcel

**Example 4** Make $t$ the subject of the formula $r = \dfrac{3t+5}{t-1}$.

| | |
|---|---|
| $r = \dfrac{3t+5}{t-1}$ | **1** Remove the fraction first by multiplying throughout by $t - 1$. |
| $r(t - 1) = 3t + 5$ | **2** Expand the brackets. |
| $rt - r = 3t + 5$ | **3** Get the terms containing $t$ on one side and everything else on the other side. |
| $rt - 3t = 5 + r$ | |
| $t(r - 3) = 5 + r$ | **4** Factorise the LHS as $t$ is a common factor. |
| $t = \dfrac{5+r}{r-3}$ | **5** Divide throughout by $r - 3$. |

## Practice

Change the subject of each formula to the letter given in the brackets.

**1** $C = \pi d$ [$d$]

**2** $P = 2l + 2w$ [$w$]

**3** $D = \dfrac{S}{T}$ [$T$]

**4** $p = \dfrac{q-r}{t}$ [$t$]

**5** $u = at - \dfrac{1}{2}t$ [$t$]

**6** $V = ax + 4x$ [$x$]

**7** $\dfrac{y-7x}{2} = \dfrac{7-2y}{3}$ [$y$]

**8** $x = \dfrac{2a-1}{3-a}$ [$a$]

**9** $x = \dfrac{b-c}{d}$ [$d$]

**10** $h = \dfrac{7g-9}{2+g}$ [$g$]

**11** $e(9 + x) = 2e + 1$ [$e$]

**12** $y = \dfrac{2x+3}{4-x}$ [$x$]

**13** Make $r$ the subject of the following formulae.

**a** $A = \pi r^2$ **b** $V = \dfrac{4}{3}\pi r^3$ **c** $P = \pi r + 2r$ **d** $V = \dfrac{2}{3}\pi r^2 h$

**14** Make $x$ the subject of the following formulae.

**a** $\dfrac{xy}{z} = \dfrac{ab}{cd}$ **b** $\dfrac{4\pi cx}{d} = \dfrac{3z}{py^2}$

**15** Make sin $B$ the subject of the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

**16** Make cos $B$ the subject of the formula $b^2 = a^2 + c^2 - 2ac \cos B$.

## Extend

**17** Make $x$ the subject of the following equations.

**a** $\dfrac{p}{q}(sx+t) = x-1$ **b** $\dfrac{p}{q}(ax+2y) = \dfrac{3p}{q^2}(x-y)$

Pearson

## Answers

**1** $d = \dfrac{C}{\pi}$

**2** $w = \dfrac{P-2l}{2}$

**3** $T = \dfrac{S}{D}$

**4** $t = \dfrac{q-r}{p}$

**5** $t = \dfrac{2u}{2a-1}$

**6** $x = \dfrac{V}{a+4}$

**7** $y = 2 + 3x$

**8** $a = \dfrac{3x+1}{x+2}$

**9** $d = \dfrac{b-c}{x}$

**10** $g = \dfrac{2h+9}{7-h}$

**11** $e = \dfrac{1}{x+7}$

**12** $x = \dfrac{4y-3}{2+y}$

**13** **a** $r = \sqrt{\dfrac{A}{\pi}}$ **b** $r = \sqrt[3]{\dfrac{3V}{4\pi}}$

**c** $r = \dfrac{P}{\pi+2}$ **d** $r = \sqrt{\dfrac{3V}{2\pi h}}$

**14** **a** $x = \dfrac{abz}{cdy}$ **b** $x = \dfrac{3dz}{4\pi cpy^2}$

**15** $\sin B = \dfrac{b\sin A}{a}$

**16** $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

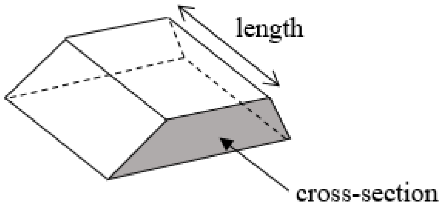
**17** **a** $x = \dfrac{q+pt}{q-ps}$ **b** $x = \dfrac{3py+2pqy}{3p-apq} = \dfrac{y(3+2q)}{3-aq}$

# Volume and surface area of 3D shapes

> **A LEVEL LINKS**
> **Scheme of work:** 6b. Gradients, tangents, normals, maxima and minima

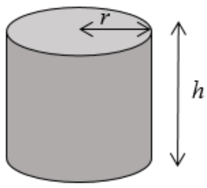
## Key points

- Volume of a prism = cross-sectional area × length.
- The surface area of a 3D shape is the total area of all its faces.

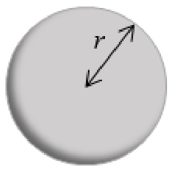



- Volume of a pyramid = $\frac{1}{3}$ × area of base × vertical height.

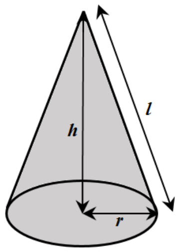
- Volume of a cylinder = $\pi r^2 h$
- Total surface area of a cylinder = $2\pi r^2 + 2\pi rh$




- Volume of a sphere = $\frac{4}{3}\pi r^3$
- Surface area of a sphere = $4\pi r^2$

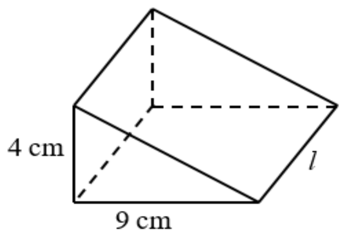



- Volume of a cone = $\frac{1}{3}\pi r^2 h$
- Total surface area of a cone = $\pi rl + \pi r^2$



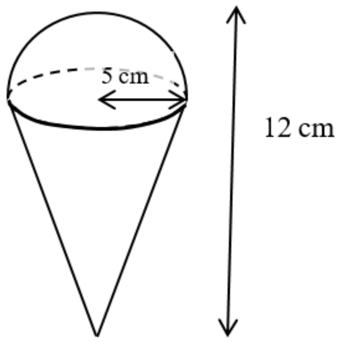

## Examples

**Example 1** The triangular prism has volume 504 cm³.
Work out its length.




| | |
|---|---|
| $V = \frac{1}{2}bhl$ | **1** Write out the formula for the volume of a triangular prism. |
| $504 = \frac{1}{2} \times 9 \times 4 \times l$ | **2** Substitute known values into the formula. |
| $504 = 18 \times l$ | **3** Simplify |
| $l = 504 \div 18$<br>$= 28$ cm | **4** Rearrange to work out $l$.<br>**5** Remember the units. |

**Example 2** Calculate the volume of the 3D solid.
Give your answer in terms of $\pi$.

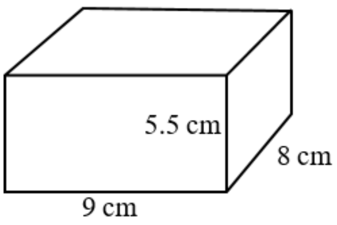



| | |
|---|---|
| Total volume = volume of hemisphere<br>+ Volume of cone<br>$= \frac{1}{2}$ of $\frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$ | **1** The solid is made up of a hemisphere radius 5 cm and a cone with radius 5 cm and height $12 - 5 = 7$ cm. |
| Total volume $= \frac{1}{2} \times \frac{4}{3} \times \pi \times 5^3$<br>$+ \frac{1}{3} \times \pi \times 5^2 \times 7$ | **2** Substitute the measurements into the formula for the total volume. |
| $= \frac{425}{3}\pi$ cm$^3$ | **3** Remember the units. |

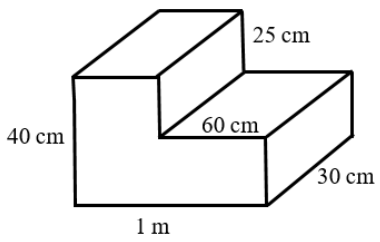
## Practice

**1** Work out the volume of each solid.
Leave your answers in terms of $\pi$ where appropriate.

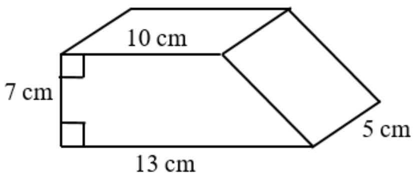
**a**



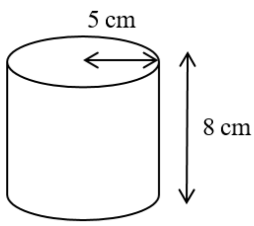

**b**



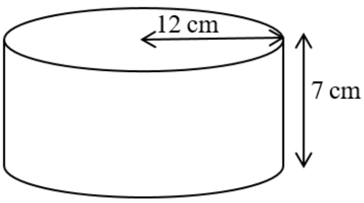

**c**




**d**




**e**

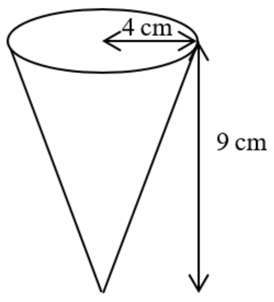



**f** a sphere with radius 7 cm

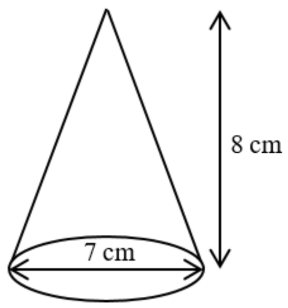
**g** a sphere with diameter 9 cm

**h** a hemisphere with radius 3 cm

**i**

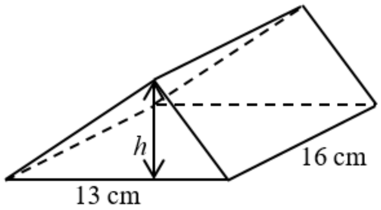



**j**




**2** A cuboid has width 9.5 cm, height 8 cm and volume 1292 cm$^3$.
Work out its length.

**3** The triangular prism has volume 1768 cm$^3$.
Work out its height.




## Extend

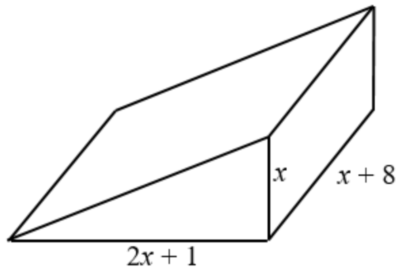
**4** The diagram shows a solid triangular prism.

All the measurements are in centimetres.

The volume of the prism is $V$ cm$^3$.

Find a formula for $V$ in terms of $x$.

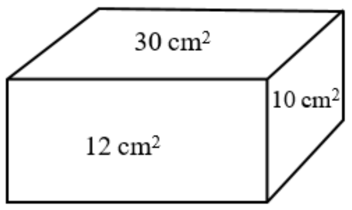
Give your answer in simplified form.




**5** The diagram shows the area of each of three faces of a cuboid.

The length of each edge of the cuboid is a whole number of centimetres.

Work out the volume of the cuboid.



**6** The diagram shows a large catering size tin of beans in the shape of a cylinder.

The tin has a radius of 8 cm and a height of 15 cm.

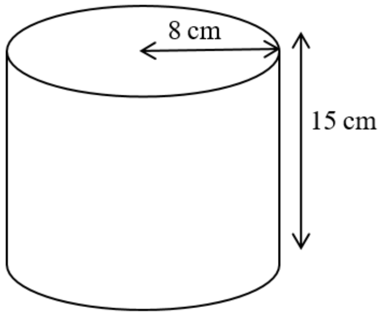
A company wants to make a new size of tin.

The new tin will have a radius of 6.7 cm.

It will have the same volume as the large tin.

Calculate the height of the new tin.

Give your answer correct to one decimal place.




**7** The diagram shows a sphere and a solid cylinder.

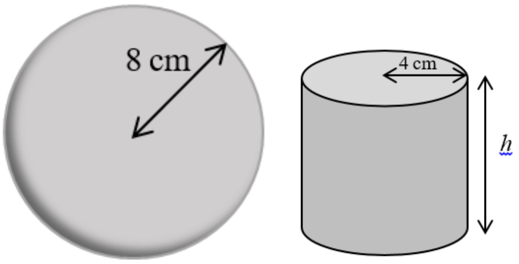
The sphere has radius 8 cm.

The solid cylinder has a base radius of 4 cm and a height of $h$ cm.

The total surface area of the cylinder is half the total surface area of the sphere.

Work out the ratio of the volume of the sphere to the volume of the cylinder.

Give your answer in its simplest form.

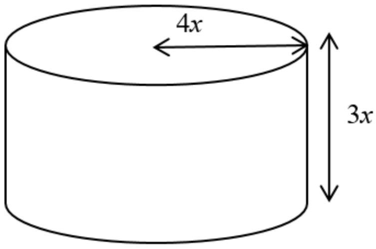



**8** The diagram shows a solid metal cylinder.

The cylinder has base radius $4x$ and height $3x$.

The cylinder is melted down and made into a sphere of radius $r$.

Find an expression for $r$ in terms of $x$.



# Answers

**1** **a** $V = 396\text{ cm}^3$

**b** $V = 75\,000\text{ cm}^3$

**c** $V = 402.5\text{ cm}^3$

**d** $V = 200\pi\text{ cm}^3$

**e** $V = 1008\pi\text{ cm}^3$

**f** $V = \frac{1372}{3}\pi\text{ cm}^3$

**g** $V = 121.5\pi\text{ cm}^3$

**h** $V = 18\pi\text{ cm}^3$

**i** $V = 48\pi\text{ cm}^3$

**j** $V = \frac{98}{3}\pi\text{ cm}^3$

**2** 17 cm

**3** 17 cm

**4** $V = x^3 + \frac{17}{2}x^2 + 4x$

**5** $60\text{ cm}^3$

**6** 21.4 cm

**7** 32 : 9

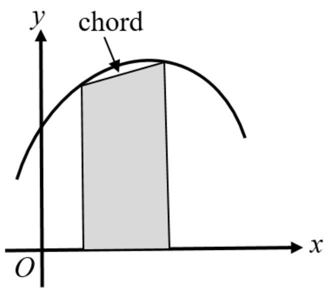
**8** $r = \sqrt[3]{36x}$

# Area under a graph

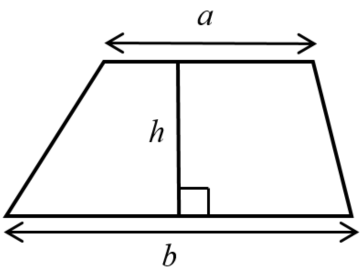
**A LEVEL LINKS**
**Scheme of work:** 7b. Definite integrals and areas under curves

## Key points

- To estimate the area under a curve, draw a chord between the two points you are finding the area between and straight lines down to the horizontal axis to create a trapezium.
  The area of the trapezium is an approximation for the area under a curve.

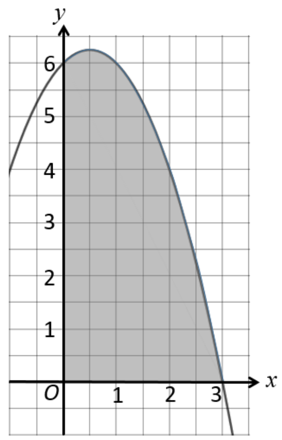



- The area of a trapezium = $\frac{1}{2}h(a+b)$




## Examples

**Example 1** Estimate the area of the region between the curve $y = (3 - x)(2 + x)$ and the $x$-axis from $x = 0$ to $x = 3$. Use three strips of width 1 unit.




| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y = (3 - x)(2 + x)$ | 6 | 6 | 4 | 0 |

Trapezium 1:
$a_1 = 6 - 0 = 6$, $b_1 = 6 - 0 = 6$
Trapezium 2:
$a_2 = 6 - 0 = 6$, $b_2 = 4 - 0 = 4$
Trapezium 3:
$a_3 = 4 - 0 = 4$, $a_3 = 0 - 0 = 0$

**1** Use a table to record the value of $y$ on the curve for each value of $x$.

**2** Work out the dimensions of each trapezium. The distances between the $y$-values on the curve and the $x$-axis give the values for $a$.

*(continued on next page)*

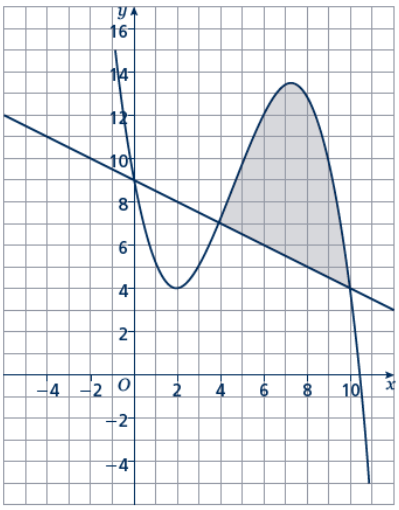
$$\frac{1}{2}h(a_1+b_1)=\frac{1}{2}\times 1(6+6)=6$$
$$\frac{1}{2}h(a_2+b_2)=\frac{1}{2}\times 1(6+4)=5$$
$$\frac{1}{2}h(a_3+b_3)=\frac{1}{2}\times 1(4+0)=2$$

Area = 6 + 5 + 2 = 13 units$^2$

**3** Work out the area of each trapezium. $h$ = 1 since the width of each trapezium is 1 unit.

**4** Work out the total area. Remember to give units with your answer.

**Example 2** Estimate the shaded area.
Use three strips of width 2 units.




| $x$ | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| $y$ | 7 | 12 | 13 | 4 |

| $x$ | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| $y$ | 7 | 6 | 5 | 4 |

Trapezium 1:
$a_1 = 7-7=0$, $b_1 = 12-6=6$
Trapezium 2:
$a_2 = 12-6=6$, $b_2 = 13-5=8$
Trapezium 3:
$a_3 = 13-5=8$, $a_3 = 4-4=0$

$$\frac{1}{2}h(a_1+b_1)=\frac{1}{2}\times 2(0+6)=6$$
$$\frac{1}{2}h(a_2+b_2)=\frac{1}{2}\times 2(6+8)=14$$
$$\frac{1}{2}h(a_3+b_3)=\frac{1}{2}\times 2(8+0)=8$$

Area = 6 + 14 + 8 = 28 units$^2$

**1** Use a table to record $y$ on the curve for each value of $x$.

**2** Use a table to record $y$ on the straight line for each value of $x$.

**3** Work out the dimensions of each trapezium. The distances between the $y$-values on the curve and the $y$-values on the straight line give the values for $a$.

**4** Work out the area of each trapezium. $h$ = 2 since the width of each trapezium is 2 units.

**5** Work out the total area. Remember to give units with your answer.

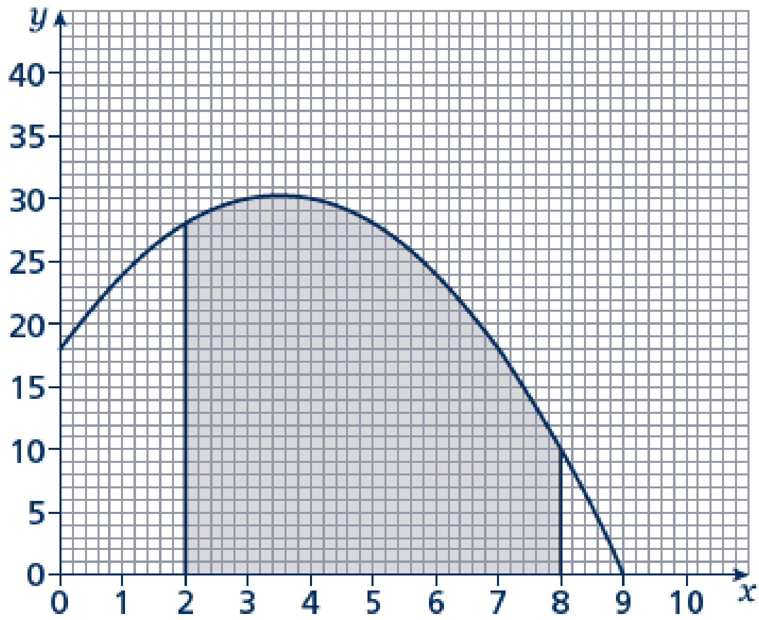
## Practice

**Hint:**

For a full answer, remember to include 'units$^2$'.

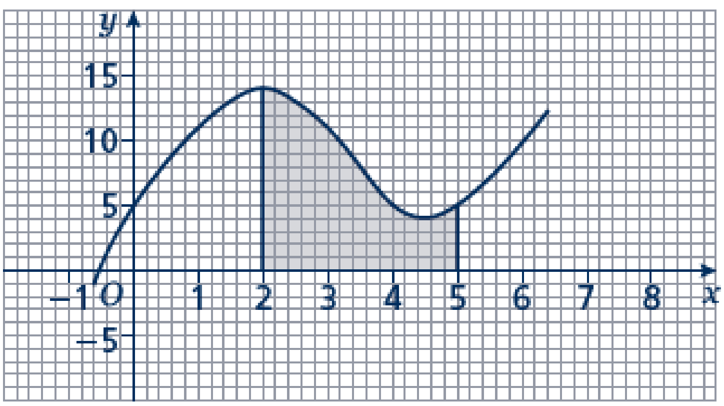
**1** Estimate the area of the region between the curve $y = (5 - x)(x + 2)$ and the $x$-axis from $x = 1$ to $x = 5$.
Use four strips of width 1 unit.

**2** Estimate the shaded area shown on the axes.
Use six strips of width 1 unit.



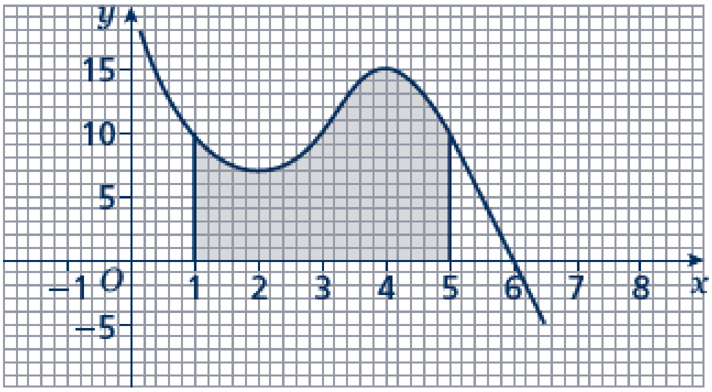

**3** Estimate the area of the region between the curve $y = x^2 - 8x + 18$ and the $x$-axis from $x = 2$ to $x = 6$.
Use four strips of width 1 unit.

**4** Estimate the shaded area.
Use six strips of width $\frac{1}{2}$ unit.



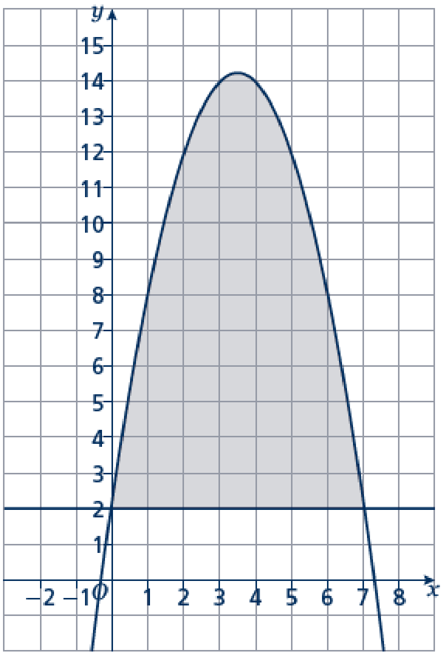
**5** Estimate the area of the region between the curve $y = -x^2 - 4x + 5$ and the $x$-axis from $x = -5$ to $x = 1$.
Use six strips of width 1 unit.

**6** Estimate the shaded area.
Use four strips of equal width.



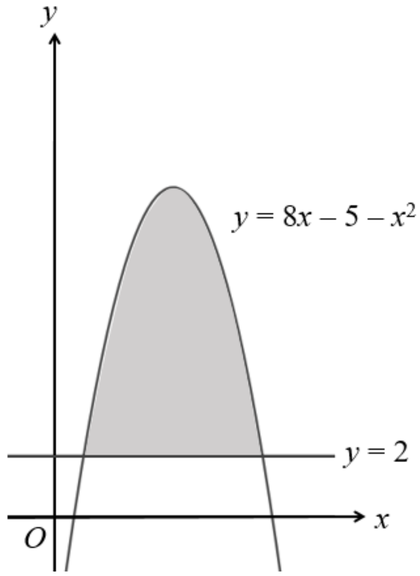

**7** Estimate the area of the region between the curve $y = -x^2 + 2x + 15$ and the $x$-axis from $x = 2$ to $x = 5$.
Use six strips of equal width.

**8** Estimate the shaded area.
Use seven strips of equal width.

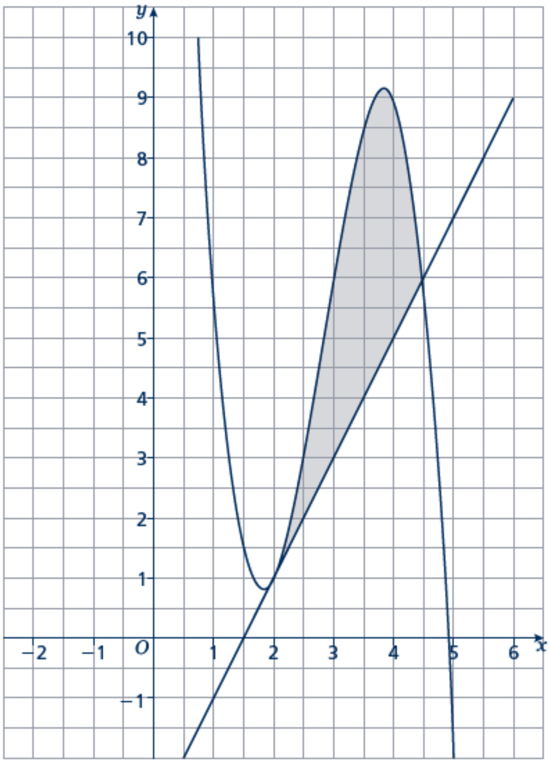


## Extend

**9** The curve $y = 8x - 5 - x^2$ and the line $y = 2$ are shown in the sketch.
Estimate the shaded area using six strips of equal width.




**10** Estimate the shaded area using five strips of equal



## Answers

**1** 34 units$^2$

**2** 149 units$^2$

**3** 14 units$^2$

**4** $25\frac{1}{4}$ units$^2$

**5** 35 units$^2$

**6** 42 units$^2$

**7** $26\frac{7}{8}$ units$^2$

**8** 56 units$^2$

**9** 35 units$^2$

**10** $6\frac{1}{4}$ units$^2$